Math 241 Calculus III (Prof. Honold) **Midterm Exam 3** 2023/12/20

Name: __________________ Student No.: __________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Let Q be the (solid) rectangle in $\mathbb{R}^2$ with vertices $(1,1)$, $(3,1)$, $(1,4)$, $(3,4)$. The integral $\int_Q xy(2x+3y)\,\mathrm{d}^2(x,y)$ equals

   ☐ 344 ☐ 363 ■ 382 ☐ 401 ☐ 420

2. With $D = \{(x,y) \in \mathbb{R}^2; x^2+4y^2 \leq 16, x \geq 0, y \geq 0\}$, the value of $\int_D x^2y\,\mathrm{d}^2(x,y)$ is contained in

   ☐ $[0,5)$ ☐ $[5,10)$ ☐ $[10,15)$ ■ $[15,20)$ ☐ $[20,+\infty)$

3. Let S be the region in $\mathbb{R}^2$ that lies above the x-axis and below the line $y = x$. The integral $\int_S \mathrm{e}^{-x^2-y^2}\,\mathrm{d}^2(x,y)$ has the value

   ■ $\pi/8$ ☐ $\pi/4$ ☐ $\pi/2$ ☐ π ☐ $+\infty$

4. For $I(t) = \int_0^\infty \frac{\ln(x^2+t)}{x^2+1}\,\mathrm{d}x$ the derivative $I'(1)$ is equal to

   ■ $\pi/4$ ☐ $\pi/2$ ☐ π ☐ 2π ☐ 4π

5. The limit $\lim_{n\to\infty} \int_0^{\pi/2} \sqrt[n]{\sin(x/n)}\,\mathrm{d}x$ is equal to

   ☐ 0 ☐ $\pi/4$ ■ $\pi/2$ ☐ π ☐ $+\infty$

6. The function $f(x,y) = x^3+y^3-2x^2+2xy-y^2+3$, $(x,y) \in \mathbb{R}^2$ has in $(0,0)$

   ☐ a local minimum ■ a local maximum ☐ a saddle point

   ☐ a non-critical point ☐ none of the foregoing

7. The number of critical points of $g(x,y) = xy(1-x^2-y^2)$, $(x,y) \in \mathbb{R}^2$ is

   ☐ 1 ☐ 3 ☐ 5 ☐ 7 ■ 9

8. Let E be the tangent plane to the surface $xyz+6=0$ in $(1,-2,3)$. Which of the following points minimizes the distance to E ?

   ☐ $(0,0,0)$ ☐ $(1,1,1)$ ☐ $(-1,1,1)$ ■ $(1,-1,1)$ ☐ $(1,1,-1)$

9. The function $x = g(y,z)$ implicitly defined by the equation $x\sin y + y\sin z + z\sin x = 0$ and $g(0,\pi/2) = 0$ has $\nabla g(0,\pi/2)$ equal to

   ☐ $(0,0)$ ☐ $(0,-2/\pi)$ ☐ $(2/\pi,0)$ ■ $(-2/\pi,0)$ ☐ $(0,2/\pi)$

10. The line integral of $y^2\,\mathrm{d}x + \mathrm{d}y$ along the curve $\gamma_\alpha(t) = (t,t^\alpha)$, $t \in [0,1]$ equals $\frac{2024}{2023}$ for

    ☐ $\alpha = 1000$ ☐ $\alpha = 1010$ ■ $\alpha = 1011$ ☐ $\alpha = 2020$ ☐ $\alpha = 2022$

Time allowed: 45 min CLOSED BOOK ***Good luck!***

Notes

Green boxes indicate the correct solutions and red boxes (if any) the most frequently made errors. This time Groups A and B were completely identical.

1 Since $Q=[1,3]\times[1,4]$, we have

$$\begin{aligned}\int_Q xy(ax+by)\,\mathrm{d}^2(x,y) &= a\int_Q x^2y\,\mathrm{d}^2(x,y)+b\int_Q xy^2\,\mathrm{d}^2(x,y)\\ &= a\int_1^3 x^2\,\mathrm{d}x\int_1^4 y\,\mathrm{d}y+b\int_1^3 x\,\mathrm{d}x\int_1^4 y^2\,\mathrm{d}y\\ &= a\,\frac{3^3-1^3}{3}\,\frac{4^2-1^2}{2}+\frac{3^2-1^2}{2}\,\frac{4^3-1^3}{3}\\ &= 65\,a+84\,b=65(a+b)+19\,b.\end{aligned}$$

Since $a+b=5$ in both groups, the correct answer is

$$325+19\,b=\begin{cases}325+19\cdot 3=382 & \text{in Group A,}\\ 325+19\cdot 2=363 & \text{in Group B.}\end{cases}$$

2 Using the polar-like coordinates $x=r\cos t$, $y=(r/2)\sin t$, $\frac{\partial(x,y)}{\partial(r,t)}=\begin{pmatrix}\cos t & -r\sin t\\ (1/2)\sin t & (r/2)\cos t\end{pmatrix}$, which has determinant $r/2$, one obtains

$$\begin{aligned}\int_D xy\,\mathrm{d}^2(x,y) &= \int_{\substack{0<r<4\\ 0<\theta<\pi/2}}(r\cos t)^2(r/2)\sin t(r/2)\,\mathrm{d}^2(r,t)\\ &= \frac{1}{4}\int_0^4 r^4\,\mathrm{d}r\int_0^{\pi/2}\cos^2 t\sin t\,\mathrm{d}t=\frac{4^5}{20}\left[-\frac{1}{3}\cos^2 t\right]_0^{\pi/2}=\frac{4^5}{60}=\frac{256}{15}.\end{aligned}$$

Thus the correct answer is (D).

3 The function $(x,y)\mapsto \mathrm{e}^{-x^2-y^2}$ is symmetric w.r.t. the lines $x=0$ and $y=x$. It follows that its integral over each of the 8 sectors $(k-1)\pi/4\le\theta\le k\pi/4$, $k=1,2,3,4,5,6,7,8$, is the same. Since $\int_{\mathbb{R}^2}\mathrm{e}^{-x^2-y^2}\,\mathrm{d}^2(x,y)=\pi$, as shown in the lecture, the correct answer must be (A). Of course one can also compute the integral directly using polar coordinates.

4 In the lecture it was shown that $I(t)$, which is defined for $t\ge 0$, can be differentiated under the integral sign for $t>0$. This gives

$$\begin{aligned}I'(t) &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\frac{\ln(x^2+t)}{x^2+1}\,\mathrm{d}x=\int_0^\infty\frac{\mathrm{d}x}{(x^2+1)(x^2+t)},\\ I'(1) &= \int_0^\infty\frac{\mathrm{d}x}{(x^2+1)^2}.\end{aligned}$$

This integral can be evaluated using integration by parts and has the value $\pi/4$. If you don't remember how to do this, observe that $\frac{1}{(x^2+1)^2}<\frac{1}{x^2+1}$ for $x>0$, and hence that the value of this integral must be smaller than $\int_0^\infty\frac{\mathrm{d}x}{x^2+1}=[\arctan x]_0^\infty=\pi/2$. This leaves only (A) as possible answer.
Alternatively, if you remember the result $I(t)=\pi\log\left(\sqrt{t}+1\right)$ from the lecture, use this to compute $I'(1)=\left.\frac{\pi}{2\sqrt{t}\left(\sqrt{t}+1\right)}\right|_{t=1}=\pi/4$.

5 We show the solution for Group A. Since $\sin(x/n)=(x/n)\cos\xi$ with $\xi\in(0,x/n)$ and $\lim_{n\to\infty}\sqrt[n]{x}=\lim_{n\to\infty}\sqrt[n]{n}=1$, we obtain $\sqrt[n]{\sin(x/n)}\to 1$ for $n\to\infty$ at all points $x\in(0,\pi/2]$. (For $x=0$ the limit is 0.) Since $0 \le \sqrt[n]{\sin(x/n)} \le 1$ and the constant function 1 is integrable over $[0,\pi/2]$, we can apply Lebesgue's Dominated Convergence Theorem to conclude

$$\lim_{n\to\infty} \int_0^{\pi/2} \sqrt[n]{\sin(x/n)}\,\mathrm{d}x = \int_0^{\pi/2} \lim_{n\to\infty} \left(\sqrt[n]{\sin(x/n)}\right) \mathrm{d}x = \int_0^{\pi/2} 1\,\mathrm{d}x = \pi/2.$$

In Group B the integration is over $[0,\pi]$, and hence the correct answer (derived in the same way) is π.

6 Use

$$\mathbf{H}_f(x,y) = \begin{pmatrix} 6x-4 & 2 \\ 2 & 6y-2 \end{pmatrix}, \quad \mathbf{H}_f(0,0) = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix}, \quad \det \mathbf{H}_f(0,0) = 4 > 0,$$

or observe that the Hesse quadratic form of f is a positive multiple of $-2x^2 + 2xy - y^2 = -2(x+y/2)^2 - y^2/2$, which is negative definite.

7 The 0-contour is the union of the lines $x = 0$, $y = 0$, and the unit circle. The five intersection points $(0,0)$, $(\pm1,\pm1)$, must be critical points, because the 0-contour isn't smooth there. Moreover, on each of the 4 quarter disks determined by the 0-contour the function g, which is continuous, attains a maximum. Since g is positive in the interior of the quarter disk, the maximum can't be on the boundary and hence must be a critical point. Thus g has at least 9 critical points, so that the correct answer must be (E)

8 The tangent plane to $xyz + 6 = 0$ in (x_0,y_0,z_0) has equation $y_0z_0(x-x_0) + x_0z_0(y-y_0) + x_0y_0(z-z_0) = 0$. Plugging in $(x_0,y_0,z_0) = (1,-2,3)$ gives $-6(x-1)+3(y+2)-2(z-3) = 0$, or $6x - 3y + 2z = 18$ as an equation for E. With $\mathbf{n} = (6,-3,2)$ and $\mathbf{p} \in E$ the distance from $\mathbf{b}$ to E is

$$|\mathrm{proj}_{\mathbf{n}}(\mathbf{b}-\mathbf{p})| = \left| \frac{(\mathbf{b}-\mathbf{p})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} \mathbf{n} \right| = \frac{|\mathbf{b}\cdot\mathbf{n} - \mathbf{p}\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{|6b_1 - 3b_2 + 2b_3 - 18|}{7}.$$

It is minimized for the point $(1,-1,1)$, and the minimal distance is 1 (coincidence?).

9 For $F(x,y,z) = x\sin y + y\sin z + z\sin x$ we have $F(0,0,\pi/2) = 0$, $F_x = \sin y + z\cos x$, $F_x(0,0,\pi/2) = \pi/2 \neq 0$, so that $g(y,z)$ is well-defined in a neighborhood of $(0,\pi/2)$. The formulas for implicit differentiation yield $g_y = -F_y/F_x = -\frac{x\cos y + \sin z}{\sin y + z\cos x}$, $g_z = -F_z/F_x = -\frac{y\cos z + \sin x}{\sin y + z\cos x}$, and hence

$$\nabla g(0,\pi/2) = \left(-\frac{F_y(0,0,\pi/2)}{F_x(0,0,\pi/2)}, -\frac{F_z(0,0,\pi/2)}{F_x(0,0,\pi/2)} \right) = (-2/\pi, 0).$$

10 As shown in the lecture (on the last slide shown on Mon Dec 18) the line integral along γ_α has the value $1 + \frac{1}{2\alpha+1} = \frac{2\alpha+2}{2\alpha+1}$. Hence the correct answer is (C).