Math 241 Calculus III (Prof. Honold) **Midterm Exam 3** 2023/05/15

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The line integral of $y\,\mathrm{d}x + x\,\mathrm{d}y$ along the curve $\gamma(t) = \big(t^2\cos t, t\cos^2 t\big)$, $t \in [0,\pi]$ equals

   ☐ π^3 ☐ π^2 ☐ $-\pi^2$ ☑ $-\pi^3$ ☐ 0

2. With $D = \big\{(x,y) \in \mathbb{R}^2; (x-3)^2 + y^2 \le 9, y \ge 0\big\}$, the integral $\int_D xy\,\mathrm{d}^2(x,y)$ is equal to

   ☐ 162 ☐ 9 ☐ 27 ☐ 81 ☑ 54

3. Let Δ be the (solid) triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,1)$, $(1,-1)$. The integral $\displaystyle\int_\Delta xy(x-y)\,\mathrm{d}^2(x,y)$ equals

   ☑ $-\frac{2}{15}$ ☐ $-\frac{1}{15}$ ☐ 0 ☐ $\frac{1}{15}$ ☐ $\frac{2}{15}$

4. The centroid ("center of mass") of the body $B_a = \big\{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 \le z \le a\big\}$ is in $(0,0,1)$ for

   ☐ $a = 2$ ☐ $a = 1$ ☐ $a = 4/3$ ☐ $a = 3$ ☑ $a = 3/2$

5. For $F(a) = \displaystyle\int_2^4 \frac{x^a}{\ln x}\,\mathrm{d}x$ the derivative $F'(1)$ is equal to

   ☐ $x/\ln x$ ☐ $-1/(2\ln 2)$ ☐ 3 ☑ 6 ☐ 1

6. The function $f(x,y) = (x^2 - y)(x - y^2)$ has in $(1,1)$

   ☐ a local minimum ☐ a local maximum ☑ a saddle point
   ☐ a singularity ☐ none of the foregoing

7. The tangent plane to the graph of $f(x,y) = x^2 - 4y^2$ in the point $(1,2,z_0)$ contains the point $(0,0,c)$ for

   ☐ $c = 10$ ☑ $c = 15$ ☐ $c = 20$ ☐ $c = 25$ ☐ $c = 30$

8. The function $y = g(x,z)$ implicitly defined by the equation $x^3 + y^3 + z^3 + xyz = 0$ and $g(1,-1) = 1$ has $\nabla g(1,-1)$ equal to

   ☐ $(1,-2)$ ☐ $(-1,2)$ ☐ $(1,2)$ ☑ $(-1,-2)$ ☐ $(-1,-\frac{1}{2})$

9. The optimization problem "minimize $x^2 + y^2 + z^2$ subject to $x^2 - yz = 3$" has the Lagrange multiplier

   ☐ $\lambda = -4$ ☐ $\lambda = -2$ ☐ $\lambda = 0$ ☑ $\lambda = 2$ ☐ $\lambda = 4$

10. The quadric surface in $\mathbb{R}^3$ with equation $x^2 - y^2 + z^2 + 4xz + 2yz = 1$ is a

    ☐ ellipsoid ☐ hyperboloid of 1 sheet ☑ hyperboloid of 2 sheets
    ☐ elliptic paraboloid ☐ hyperbolic paraboloid

Time allowed: 45 min CLOSED BOOK ***Good luck!***

Notes

Green boxes indicate the correct solutions and red boxes (if any) the most frequently made errors. This time Groups A and B were completely identical.

1 A smart way to answer this question is to observe that $y\,\mathrm{d}x + x\,\mathrm{d}y = \mathrm{d}(xy)$, the differential of the function $f(x,y) = xy$, so that the Fundamental Theorem for Line Integrals yields

$$\int_\gamma y\,\mathrm{d}x + x\,\mathrm{d}y = \left(\pi^2\cos\pi\right)\left(\pi\cos^2\pi\right) - 0\cdot 0 = -\pi^3.$$

2 Using polar coordinates $x = 3 + r\cos t$, $y = r\sin t$ one obtains

$$\begin{aligned}\int_D xy\,\mathrm{d}^2(x,y) &= \int_{\substack{0<r<3\\0<\theta<\pi}} (3 + r\cos\theta)r\sin\theta\, r\,\mathrm{d}^2(r,\theta)\\ &= 3\int_0^3 r^2\,\mathrm{d}r\int_0^\pi \sin\theta\,\mathrm{d}\theta + \int_0^3 r^3\,\mathrm{d}r\int_0^\pi \cos\theta\sin\theta\,\mathrm{d}\theta = 3\left[\frac{r^3}{3}\right]_0^3 2 = 54,\end{aligned}$$

since $\int_0^\pi \cos\theta\sin\theta\,\mathrm{d}\theta = 0$ by symmetry.

3 The triangle is bounded by the lines $y = \pm x$ and $x = 1$.

$$\begin{aligned}\implies \int_\Delta xy(x-y)\,\mathrm{d}^2(x,y) &= \int_\Delta x^2y\,\mathrm{d}^2(x,y) - \int_\Delta xy^2\,\mathrm{d}^2(x,y)\\ &= -\int_\Delta xy^2\,\mathrm{d}^2(x,y) && \text{(by symmetry)}\\ &= -2\int_{x=0}^1\int_{y=0}^x xy^2\,\mathrm{d}y\,\mathrm{d}x && \text{(symmetry, Fubini)}\\ &= -2\int_0^1 x\left[\frac{y^3}{3}\right]_0^x\mathrm{d}x = -\frac{2}{3}\int_0^1 x^4\,\mathrm{d}x = -\frac{2}{15}.\end{aligned}$$

4 The z-coordinate of the centroid is

$$\frac{\int_{B_a} z\,\mathrm{d}^3(x,y,z)}{\int_{B_a} 1\,\mathrm{d}^3(x,y,z)} = \frac{\int_0^a z(z\pi)\,\mathrm{d}z}{\int_0^a z\pi\,\mathrm{d}z} = \frac{a^3/3}{a^2/2} = \frac{2}{3}a,$$

since the z-section of B_a is a disk of radius $\sqrt{z}$ for $0 \le z \le a$ and empty for $z < 0$ or $z > a$.

5 Since the integration is over a compact interval and the integrand $f(x,a) = \frac{x^a}{\ln x}$ is a continuous 2-variable function, we can differentiate under the integral sign to obtain

$$F'(a) = \int_2^4 \frac{\partial}{\partial a}\frac{x^a}{\ln x}\,\mathrm{d}x = \int_2^4 x^a\,\mathrm{d}x = \left[\frac{x^{a+1}}{a+1}\right]_2^4 = \frac{4^{a+1} - 2^{a+1}}{a+1},$$

and in particular $F'(1) = 6$.

6 Use

$$\mathbf{H}_f(x,y) = \begin{pmatrix} -2y^2 + 6x & -4xy - 1\\ -4xy - 1 & -2x^2 + 6y\end{pmatrix},\quad \mathbf{H}_f(1,1) = \begin{pmatrix} 4 & -5\\ -5 & 4\end{pmatrix}\quad \det\mathbf{H}_f(1,1) = -9 < 0,$$

or

$$\begin{aligned} f(1+h_1, 1+h_2) &= \left((1+h_1)^2 - 1 - h_2\right)\left(1 + h_1 - (1+h_2)^2\right)\\ &= \left(2h_1 - h_2 + h_1^2\right)\left(h_1 - 2h_2 - h_2^2\right)\\ &= (2h_1 - h_2)(h_1 - 2h_2) + \text{monomials of degree} \ge 3,\end{aligned}$$

which shows $\nabla f(1,1) = 0$ (no linear term) and that the Hesse quadratic form of f at $(1,1)$ is a product of two linear forms and hence indefinite.

7 The graph is the level-zero set of $F(x,y,z) = x^2 - 4y^2 - z$, which has $\nabla F(x,y,z) = (2x, -8y, -1)$, $\nabla F(1,2,-15) = (2,-16,-1)$. Hence an equation for the tangent plane is $2(x-1) - 16(y-2) - (z+15) = 0$, or $2x - 16y - z = -15$, and the tangent plane meets the z-axis in $(0,0,15)$.

8 For $F(x,y,z) = x^3 + y^3 + z^3 + xyz$ we have $F(1,1,-1) = 0$, $F_y = 3y^2 + xz$, $F_y(1,1,-1) = 2 \neq 0$, so that $g(x,z)$ is well-defined in a neighborhood of $(1,-1)$. The formulas for implicit differentiation yield $g_x = -F_x/F_y = -\frac{3x^2+yz}{3y^2+xz}$, $g_z = -F_z/F_y = -\frac{3z^2+xy}{3y^2+xz}$, and hence

$$\nabla g(1,-1) = \left(-\frac{F_x(1,1,-1)}{F_y(1,1,-1)}, -\frac{F_z(1,1,-1)}{F_y(1,1,-1)} \right) = (-1,-2).$$

9 For $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^2 - yz$ we have $\nabla f(x,y,z) = (2x, 2y, 2z)$, $\nabla g(x,y,z) = 2x, -z, -y)$, so that $\lambda \in \mathbb{R}$ is a Lagrange multiplier iff the system

$$\begin{aligned} 2x &= \lambda(2x) \\ 2y &= -\lambda z \\ 2z &= -\lambda y \\ x^2 - yz &= 3 \end{aligned}$$

has a solution (x,y,z) satisfying $g(x,y,z) = 3$ and $\nabla g(x,y,z) \neq (0,0,0)$. The 2nd and 3rd equation give $4y = -2\lambda z = \lambda^2 y$, and hence $y = 0 \vee \lambda = \pm 2$. If $y = 0$ then $z = 0$, $x = \pm\sqrt{3} \neq 0$, and from the 1st equation $\lambda = 1$, which isn't offered as answer. If $\lambda = -2$ then $x = 0$, $y = z$, which contradicts the 4th equation. For $\lambda = 2$ there are solutions, viz. $(x,y,z) = \left(0, \pm\sqrt{3}, \mp\sqrt{3}\right)$.

10 The quadric surface, call it Q, is of he form $(x,y,z)\mathbf{A}(x,y,z)^\mathsf{T} = 0$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Transforming $\mathbf{A}$ into Sylvester canonical form, we get

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow[C3=C3-2C1]{R3=R3-2R1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow[C3=C3+C2]{R3=R3+R2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$\Longrightarrow$ Q is equivalent to $x^2 - y^2 - 2z^2 = 1$, which is a hyperboloid of two sheets.