Math 241 Calculus III
Prof. Honold

Midterm Exam 2

Nov 22, 2023
18:00–19:40

Name: ______________ Student ID: ______________ Major: ______

Question 1 (ca. 8 marks)

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) \;=\; xy(6-x-y).$$

a) Which obvious symmetry property does f have? What can you conclude from this about the contours of f?

b) Describe the 0-contour of f geometrically.

c) Determine all critical points of f.

d) Make an accurate figure of the 0-contour and the 4-contour in the first quadrant $x, y > 0$ (unit length at least 1 cm). The figure should include the points on the 4-contour with a horizontal or vertical tangent, and the intersection points with the line $x = y$.

e) Determine the slope of the graph G_f at $(2,3)$ in south-eastern (SE) direction, and the maximal slope/direction of G_f at $(2,3)$.

Question 2 (ca. 6 marks)

Find the limit, if it exists, or show that the limit does not exist.

a) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\sqrt{x^2+1}-y}{x+y-1}$;

b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sin x - \sin y}{x^2-y^2}$;

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^5+y^5}{x^2y^2}$;

d) $\displaystyle\lim_{|(x,y)|\to\infty} \frac{x^3+y^3}{x^4+y^4}$.

Question 3 (ca. 4 marks)

Let $r(b,c)$ be the largest zero of x^2+bx+c, assuming $b^2-4c>0$.

a) Compute the differential $\mathrm{d}r(b,c)$, and the linear approximation to $r(b,c)$ in the point $(2,-3)$.

b) Using a), give approximate bounds for $r(b,c)$ if $|b-2| \le 0.1$ and $|c+3| \le 0.1$.

c) Using the Mean Value Theorem, make the bounds in b) rigorous.

Question 4 (ca. 6 marks)

Consider the curve C in $\mathbb{R}^3$ parametrized by

$$\gamma(t) = \left(t^3+t^2+1,\, t^2-t,\, -t^3-t-1\right), \quad t \in \mathbb{R}.$$

a) Is C contained in a plane? Justify your answer!

b) Determine the center and radius of the osculating circle and the TNB frame of C in $(1,0,-1)$.

Solutions

1 a) $f(x,y)=f(y,x)$ for $(x,y)\in\mathbb{R}^2$, implying that each contour of f is symmetric with respect to the line $y=x$ in $\mathbb{R}^2$. $\boxed{1}$

b) The 0-contour of f is the union of the three lines $x=0$, $y=0$, $x+y=6$, which form a triangle with vertices $(0,0)$, $(6,0)$, $(0,6)$. $\boxed{1}$

c) $\nabla f(x,y)=(6y-2xy-y^2,6x-x^2-2xy)=\bigl(y(6-2x-y),x(6-x-2y)\bigr)=(0,0)$
The solutions with $x=0$ or $y=0$ are $(0,0)$, $(6,0)$, $(0,6)$. $\boxed{1\frac{1}{2}}$
Other solutions must solve the linear system $2x+y=x+2y=6$, which has the unique solution $(2,2)$. $\boxed{\frac{1}{2}}$

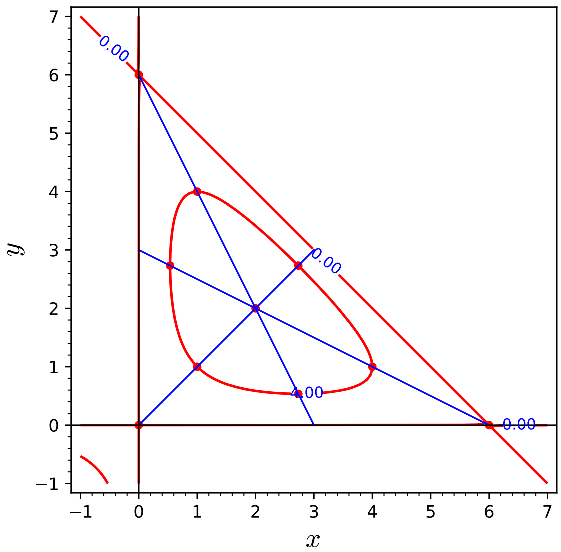
d) The 4-contour C has equation $6xy-x^2y-xy^2=4$. Points on C with $x=y$ correspond to solutions of $2x^3-6x^2+4=0$ or $x^3-3x^2+2=0$, which factors as $(x-1)(x^2-2x-2)=0$, giving the points $(1,1)$, $\bigl(1+\sqrt{3},1+\sqrt{3}\bigr)\approx(2.7,2.7)$ (and a point in the 3rd quadrant).

Horizontal tangents require $f_x=0$ (and $f_y\neq 0$), which means the intersection points of C with the line $y=6-2x$. Substituting $y=6-2x$ into the equation for C gives

$$6x(6-2x)-x^2(6-2x)-x(6-2x)^2=-2x^3+6x^2=4\quad\text{or}\quad x^3-3x^2+2=0,$$

the same equation as above. The corresponding points are $(1,4)$ and $\bigl(1+\sqrt{3},4-2\sqrt{3}\bigr)\approx(2.7,0.5)$.

By symmetry, the points with a vertical tangent are obtained by reflecting these points at the line $y=x$, i.e., $(4,1)$ and $\bigl(4-2\sqrt{3},1+\sqrt{3}\bigr)\approx(0.5,2.7)$.



$\boxed{3}$

The figure also contains the 4th critical point $(2,2)$, which is the centroid of the triangle formed by the 0-contour. The extended figure reveals the interesting fact that the points on C with a tangent parallel to one of the sides of the triangle are the intersection points with the corresponding medians of the triangle (the blue lines).

e) The slope in south-eastern direction is $\nabla f(2,3)\cdot\frac{1}{\sqrt{2}}(1,-1)=\frac{1}{\sqrt{2}}(-3,-4)\cdot(1,-1)=\frac{1}{\sqrt{2}}\approx 0.7$. $\boxed{1}$

The maximum slope is in the direction of the gradient, viz. $\nabla f(2,3)=(-3,-4)$, and has the value $|(-3,-4)|=\sqrt{3^2+4^2}=5$. $\boxed{1}$

Remarks:

$$\sum_1 = 9$$

2 a) $$\frac{\sqrt{x^2+1}-y}{x+y-1}=\frac{\left(\sqrt{x^2+1}-y\right)\left(\sqrt{x^2+1}+y\right)}{(x+y-1)\left(\sqrt{x^2+1}+y\right)}=\frac{x^2+1-y^2}{(x+y-1)\left(\sqrt{x^2+1}+y\right)}$$

Since the 2nd factor in the denominator tends to 2, the limit exists iff $\lim_{(x,y)\to(0,1)}\frac{x^2+1-y^2}{x+y-1}$ exists. But the latter function is $-(y+1)$ for $x=0$ and x for $y=1$, and approaching $(0,1)$ along the coordinate lines gives the contradiction $-2=0$.

Conclusion: The limit doesn't exist. $\boxed{1\frac{1}{2}}$

b) By the Mean Value Theorem of one-variable calculus, there exists ξ between x and y such that $\sin x-\sin y=\cos(\xi)(x-y)$. It follows that

$$\frac{\sin x-\sin y}{x^2-y^2}=\frac{\cos\xi}{x+y}\to\frac{\cos 1}{2}\quad\text{for } (x,y)\to(1,1),$$ $\boxed{1\frac{1}{2}}$

since $x\to 1$ and $y\to 1$ obviously implies $\xi\to 1$.

c) On the curve $y=x^2$ we have

$$\frac{x^5+y^5}{x^2y^2}=\frac{x^5+x^{10}}{x^6}=\frac{1}{x}+x^4\to\pm\infty+0\quad\text{for } x\to 0,$$

i.e., the limit is $+\infty$ if we approach $(0,0)$ from the right and $-\infty$ if we approach $(0,0)$ from the left.

Conclusion: The limit doesn't exist. $\boxed{1\frac{1}{2}}$

d) In polar coordinates we have

$$\frac{x^3+y^3}{x^4+y^4}=\frac{r^3\cos^3\phi+r^3\sin^3\phi}{r^4\cos^4\phi+r^4\sin^4\phi}=\frac{1}{r}\,\frac{\cos^3\phi+\sin^3\phi}{\cos^4\phi+\sin^4\phi}$$

For $r\to\infty$ the first factor tends to zero. The second factor remains bounded, since it is a continuous function on $[0,2\pi]$. (The key point is that cos and sin have no common zero and hence the denominator is well-defined for $\phi\in[0,2\pi]$.)

Conclusion: The limit is zero. $\boxed{1\frac{1}{2}}$

$$\sum_2 = 6$$

3

$$
\begin{aligned}
r(b,c) &= \frac{-b+\sqrt{b^2-4c}}{2},\\
\mathrm{d}r(b,c) &= r_b(b,c)\mathrm{d}b + r_c(b,c)\mathrm{d}c\\
&= \left(-\frac{1}{2}+\frac{b}{2\sqrt{b^2-4c}}\right)\mathrm{d}b - \frac{1}{\sqrt{b^2-4c}}\,\mathrm{d}c && \boxed{1}\\
r\left(2+\Delta b, -3+\Delta c\right) &\approx r(2,-3) + r_b(2,-3)\Delta b + r_c(2,-3)\Delta c\\
&= 1 + \left(-\frac{1}{2}+\frac{2}{2\sqrt{16}}\right)\Delta b - \frac{1}{\sqrt{16}}\,\Delta c = 1 - \frac{\Delta b}{4} - \frac{\Delta c}{4} && \boxed{1}
\end{aligned}
$$

Together with $|\Delta b| \leq 0.1$, $|\Delta c| \leq 0.1$ this gives $0.95 \leq r(b,c) \leq 1.05$ (non-rigorously). $\boxed{1}$

The Mean Value Theorem gives

$$
r\left(2+\Delta b, -3+\Delta c\right) = 1 + \left(-\frac{1}{2}+\frac{\beta}{2\sqrt{\beta^2-4\gamma}}\right)\Delta b - \frac{1}{\sqrt{\beta^2-4\gamma}}\,\Delta c
$$

with β, γ between 2 and $2+\Delta b$, respectively, -3 and $-3+\Delta c$. Together with $|\Delta b| \leq 0.1$, $|\Delta c| \leq 0.1$ this gives

$$
|r(b,c)-1| \leq \left(\frac{1}{2} - \frac{1.9}{2\sqrt{2.1^2+4\cdot 3.1}}\right)0.1 + \frac{1}{\sqrt{1.9^2+4\cdot 2.9}}\,0.1 \qquad \boxed{1}
$$

With calculator we can evaluate the right-hand side and obtain $|r(b,c)-1| \leq 0.0525$. (The right-hand side is even a rational number, viz. $\frac{839}{15990} = \frac{839}{2\cdot 3\cdot 5\cdot 13\cdot 41}$.)

Remarks: ______________________________

$$\sum_3 = 4$$

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4 a) Yes. We have

$$
\begin{aligned}
\gamma(t) &= \begin{pmatrix} t^3+t^2+1 \\ t^2-t \\ -t^3-t-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + t^2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t^3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\\
&= \begin{pmatrix} t^3+t^2+1 \\ t^2-t \\ -t^3-t-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (t^3+t)\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + (t^3+t^2)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},
\end{aligned}
$$

which is contained in the plane $E = (1,0,-1) + \mathbb{R}(0,1,1) + \mathbb{R}(1,1,0)$. $\boxed{1}$

b) Using $(1,0,-1)=\gamma(0)$, we obtain

$$\gamma'(t)=\begin{pmatrix}3\,t^2+2t\\2t-1\\-3t^2-1\end{pmatrix},$$

$$\gamma''(t)=\begin{pmatrix}6\,t+2\\2\\-6t\end{pmatrix},$$

$$\mathbf{T}(0)=\frac{\gamma'(0)}{|\gamma'(0)|}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\-1\end{pmatrix}, \qquad \boxed{1}$$

$$\lambda\,\mathbf{N}(0)=|\gamma'(0)|\,\mathbf{T}'(0)=\gamma''(0)-\frac{\gamma''(0)\cdot\gamma'(0)}{\gamma'(0)\cdot\gamma'(0)}\,\gamma'(0) \qquad (\lambda>0)$$

$$=\begin{pmatrix}2\\2\\0\end{pmatrix}-\frac{(2,2,0)\cdot(0,-1,-1)}{(0,-1,-1)\cdot(0,-1,-1)}\begin{pmatrix}0\\-1\\-1\end{pmatrix}=\begin{pmatrix}2\\2\\0\end{pmatrix}+\begin{pmatrix}0\\-1\\-1\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\mathbf{N}(0)=\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}, \qquad \boxed{1}$$

$$\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)=\frac{1}{\sqrt{12}}\begin{pmatrix}0\\-1\\-1\end{pmatrix}\begin{pmatrix}2\\1\\-1\end{pmatrix}=\frac{1}{\sqrt{12}}\begin{pmatrix}2\\-2\\2\end{pmatrix}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \qquad \boxed{1}$$

$$\kappa(0)=\frac{|\mathbf{T}'(0)|}{|\gamma'(0)|}=\frac{|\lambda\,\mathbf{N}(0)|}{|\gamma'(0)|^2}=\frac{\sqrt{6}}{2}.$$

Alternatively, compute the curvature as

$$\kappa(0)=\frac{|\gamma'(0)\times\gamma''(0)|}{|\gamma'(0)|^3}=\frac{1}{2\sqrt{2}}\left|\begin{pmatrix}0\\-1\\-1\end{pmatrix}\times\begin{pmatrix}2\\2\\0\end{pmatrix}\right|=\frac{1}{2\sqrt{2}}\left|\begin{pmatrix}2\\-2\\2\end{pmatrix}\right|=\frac{1}{\sqrt{2}}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\right|=\frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{6}}{2}.$$

$\Longrightarrow$ The radius of the osculating circle of C in $(0,3,2)$ is

$$\frac{1}{\kappa(0)}=\frac{2}{\sqrt{6}}=\frac{1}{3}\sqrt{6}, \qquad \boxed{1}$$

and the center is

$$\gamma(0)+\frac{1}{\kappa(0)}\,\mathbf{N}(0)=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}2\\1\\-1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}5\\1\\-4\end{pmatrix}. \qquad \boxed{1}$$

Remarks:

$$\sum_{4}=6$$

$$\sum_{\text{Midterm 2}}=20+5$$