Name: ____________ Student ID: ____________ Major: ______

Question 1 (ca. 11 marks)

Consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) = \frac{1}{x^2 - xy + y^2}.$$

Here $D \subseteq \mathbb{R}^2$ is the maximum possible domain for f.

a) Determine D.

b) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibilities $\pm\infty$), or show that the limit does not exist.

c) Determine all critical points of f (if any).

d) Determine the shape of the 1-contour C_1 of f and graph C_1 as accurately as possible (unit length at least 2 cm).

Hint: There are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $x^2 - xy + y^2 = \lambda_1 \left(\frac{x+y}{\sqrt{2}}\right)^2 + \lambda_2 \left(\frac{x-y}{\sqrt{2}}\right)^2$.

e) Determine the slope of the graph G_f at $(1,1)$ in the western (W) direction, and the maximal slope/direction of G_f at $(1,1)$.

f) Express $f(tx,ty)$, $t \in \mathbb{R}$, in terms of $f(x,y)$. Use the result to describe the relation between the contours of f, and sketch the k-contour of f for $k = 2$ and $k = 4$; cf. d).

Question 2 (ca. 6 marks)

Consider the differentiable map $G\colon D \to \mathbb{R}^2$, $D = \mathbb{R}^2 \setminus \bigl\{(0,0)\bigr\}$, defined by

$$G(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right).$$

a) Compute the Jacobi matrix $\mathbf{J}_G(x,y)$ and show that G is conformal.

b) Determine the G-image $S = G(\Delta)$ of the (solid) triangle Δ with vertices $(1,0)$, $(0,1)$, $(1,1)$, and graph Δ and the region S on paper (unit length at least 2 cm).

Hint: The G-images of the edges of Δ are circular arcs. Corresponding equations can be obtained by writing $G(x,y) = (u,v)$ and expressing $u^2 + v^2$ in terms of u, v. In the figure you should indicate the correspondence between edges and their G-images by using the same color (or line style such as "dashed", "dotted", etc.).

c) Is the figure obtained in b) compatible with the result in a)? Justify your answer!

Question 3 (ca. 5 marks)

Consider the curve C in $\mathbb{R}^3$ parametrized by

$$\gamma(t) = \left(t^3 - t, 2\,t^3 + 1, 3\,t - 1\right), \quad t \in \mathbb{R}.$$

a) Is C contained in a plane? Justify your answer!

b) Determine the center and radius of the osculating circle of C in $(0, 3, 2)$.

Solutions

1 a) $x^2 - xy + y^2 = \left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 = 0$ has only the trivial solution $x = y = 0$. $\Longrightarrow D = \mathbb{R}^2 \setminus \{(0,0)\}$ $\boxed{\frac{1}{2}}$

b) Using polar coordinates, we have

$$f(x,y) = f(r\cos\phi, r\sin\phi) = \frac{1}{r^2(1-\cos\phi\sin\phi)} = \frac{1}{r^2\left(1 - \frac{1}{2}\sin(2\phi)\right)}.$$

It follows that $\frac{2}{3r^2} \leq f(x,y) \leq \frac{2}{r^2}$.

The first inequality implies $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\phi, r\sin\phi) = +\infty$. $\boxed{1}$

The second inequality implies $\lim_{|(x,y)|\to\infty} f(x,y) = \lim_{r\to\infty} f(r\cos\phi, r\sin\phi) = 0$. $\boxed{1}$

c) We have

$$f_x = -\frac{2x-y}{(x^2-xy+y^2)^2}, \qquad \boxed{\tfrac{1}{2}}$$
$$f_y = -\frac{2y-x}{(x^2+xy+y^2)^2}, \qquad \boxed{\tfrac{1}{2}}$$

and $f_x = f_y = 0$ iff $2x - y = 2y - x = 0$. The only solution is $(x,y) = (0,0)$, but $(0,0) \notin D$. Hence f has no critical point. $\boxed{1}$

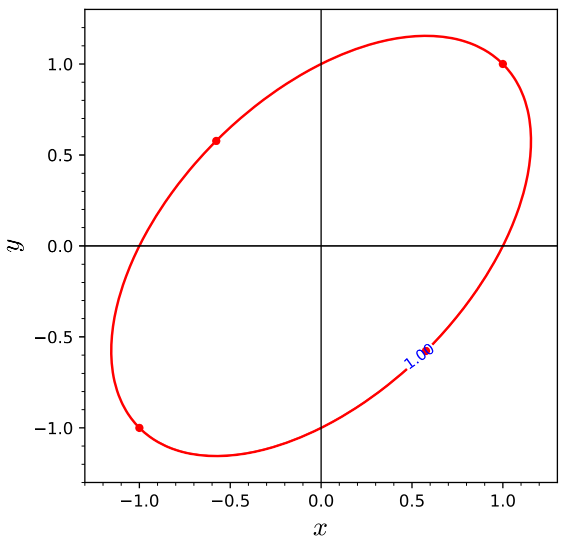
d) First we determine λ_1, λ_2 according to the hint. Expanding gives

$$x^2 - xy + y^2 = \frac{\lambda_1+\lambda_2}{2}x^2 + (\lambda_1 - \lambda_2)\,xy + \frac{\lambda_1+\lambda_2}{2}y^2$$

Equating coefficients of x^2, we obtain $\lambda_1 + \lambda_2 = 2$, $\lambda_1 - \lambda_2 = -1$, and hence $\lambda_1 = 1/2$, $\lambda_2 = 3/2$.

$$x^2 - xy + y^2 = \frac{1}{2}\left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{x-y}{\sqrt{2}}\right)^2 \qquad \boxed{1}$$

Setting $x' = \frac{x+y}{\sqrt{2}}$, $y' = \frac{x-y}{\sqrt{2}}$, the equation of the 1-contour of f becomes $x^2 - xy + y^2 = \frac{1}{2}x'^2 + \frac{3}{2}y'^2 = 1$. Since the corresponding coordinate change $\binom{x'}{y'} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\binom{x}{y}$ is a Euclidean motion (composed of a reflection at the y-axis and a 45°-degree rotation around the origin), the 1-contour is an ellipse with semi-axes $a = \sqrt{2}$, $b = \sqrt{2}/\sqrt{3}$. The vertices of the ellipse have $x' = 0$ or $y' = 0$, i.e., $x = \pm y$. Substituting this into $x^2 - xy + y^2 = 1$ gives the four points $(\pm 1, \pm 1)$, $\left(\pm\frac{1}{3}\sqrt{3}, \mp\frac{1}{3}\sqrt{3}\right)$, where either the upper or the lower signs hold.



$\boxed{2}$

e) The slope in the western direction is simply the negative of the partial derivative f_x, i.e., $-f_x(1,1) = 1$. $\boxed{\frac{1}{2}}$

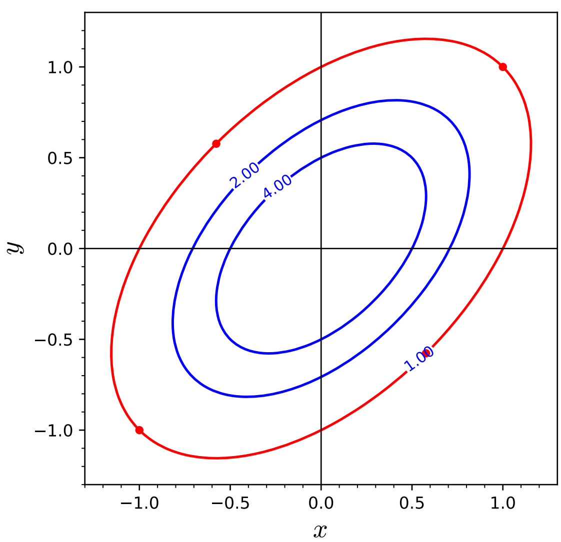
The maximum slope is in the direction of the gradient, viz. $\nabla(1,1) = (-1,-1)$ (south-west, towards the origin), and has the value $|\nabla(1,1)| = \sqrt{2}$. $\boxed{1}$

f) We have

$$f(tx,ty) = \frac{1}{(tx)^2 - (tx)(ty) + (ty)^2} = \frac{1}{t^2}\,f(x,y).$$

$\Longrightarrow C_k$ is obtained from C_1 by scaling it with the factor $1/\sqrt{k}$. $\boxed{1}$

For $k = 2, 4$ the factors are $1/\sqrt{2} \approx 0.7$ and $1/2$, respectively; cf. picture.



$\boxed{1}$

Remarks:

$$\sum_1 = 11$$

2 a) Writing $G = (u, v)^\mathsf{T}$ with $u(x, y) = \frac{x}{x^2+y^2}$, $v(x, y) = -\frac{y}{x^2+y^2}$, we have

$$\begin{aligned}\mathbf{J}_G(x, y) &= \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} 1(x^2 + y^2) - (2x)x & -2xy \\ 2xy & (-1)(x^2 + y^2) + (2y)y \end{pmatrix} \\ &= \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}. \end{aligned} \qquad \boxed{1}$$

Further we obtain

$$\begin{aligned}\mathbf{J}_G(x, y)^\mathsf{T}\mathbf{J}_G(x, y) &= \frac{1}{(x^2 + y^2)^4} \begin{pmatrix} (y^2 - x^2)^2 + 4x^2y^2 & 0 \\ 0 & (y^2 - x^2)^2 + 4x^2y^2 \end{pmatrix} \\ &= \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

$\Longrightarrow$ G is conformal in all points of D. $\boxed{1}$

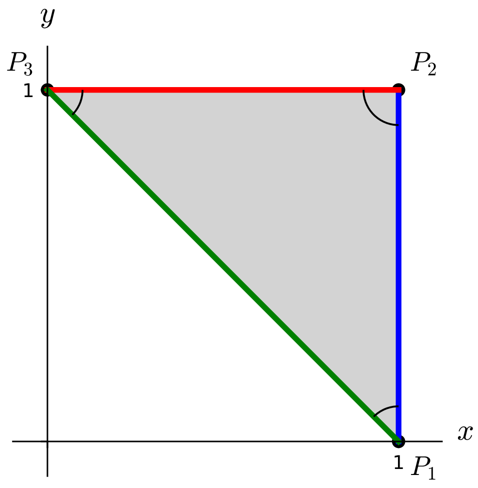
b) Writing $P_1 = (1, 0)$, $P_2 = (1, 1)$, $P_3 = (0, 1)$ for the vertices of Δ and $[P_1, P_2]$, etc., for the edges of Δ, we have $G(1, 0) = (1, 0)$, $G(0, 1) = (0, -1)$, $G(1, 1) = \left(\frac{1}{2}, -\frac{1}{2}\right)$, and further:

(i) $G(1, y) = \left(\frac{1}{1+y^2}, -\frac{y}{1+y^2}\right)$
$u^2 + v^2 = \frac{1}{1+y^2} = u \iff \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$
$\Longrightarrow G\big([P_1, P_2]\big)$ is an arc of the circle with center $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. $\boxed{\frac{1}{2}}$

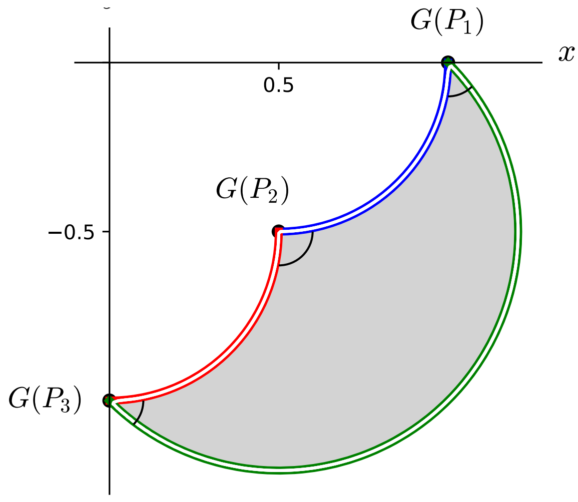
(ii) $G(x, 1) = \left(\frac{x}{1+x^2}, -\frac{1}{1+x^2}\right)$
$u^2 + v^2 = \frac{1}{1+x^2} = -v \iff u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$
$\Longrightarrow G\big([P_2, P_3]\big)$ is an arc of the circle with center $\left(0, -\frac{1}{2}\right)$ and radius $\frac{1}{2}$. $\boxed{\frac{1}{2}}$

(iii) $G(x, 1 - x) = \left(\frac{x}{x^2+(1-x)^2}, -\frac{1-x}{x^2+(1-x)^2}\right)$
$u^2 + v^2 = \frac{1}{x^2+(1-x)^2} = u - v \iff \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$
$\Longrightarrow G\big([P_3, P_1]\big)$ is an arc of the circle with center $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{2}}{2}$. $\boxed{\frac{1}{2}}$

c) Since G is conformal in P_1, P_2, P_3, the angles between the sides of Δ (45°, 90°, 45°, respectively) must be the same as the angles between their image curves, which form the boundary of $G(\Delta)$. This is also visible in the figure of $G(\Delta)$. $\boxed{1}$



(a) Δ



(b) $G(\Delta)$

$\boxed{1\frac{1}{2}}$

$\sum_2 = 6$

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3 a) Yes. We have

$$\gamma(t) = \begin{pmatrix} t^3 - t \\ 2\,t^3 + 1 \\ 3\,t - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + t^3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},$$

which is obviously contained in the plane $E = (0, 1, -1) + \mathbb{R}(-1, 0, 3) + \mathbb{R}(1, 2, 0)$. $\boxed{1}$

b) Using $(0,3,2)=\gamma(1)$, we obtain

$$\gamma'(t)=\begin{pmatrix}3\,t^2-1\\6\,t^2\\3\end{pmatrix},$$

$$\gamma''(t)=\begin{pmatrix}6\,t\\12\,t\\0\end{pmatrix},$$

$$\lambda\,\mathbf{N}(1)=|\gamma'(1)|\,\mathbf{T}'(1)=\gamma''(1)-\frac{\gamma''(1)\cdot\gamma'(1)}{\gamma'(1)\cdot\gamma'(1)}\,\gamma'(1)\qquad(\lambda>0)$$

$$=\begin{pmatrix}6\\12\\0\end{pmatrix}-\frac{(6,12,0)\cdot(2,6,3)}{(2,6,3)\cdot(2,6,3)}\begin{pmatrix}2\\6\\3\end{pmatrix}=\begin{pmatrix}6\\12\\0\end{pmatrix}-\frac{84}{49}\begin{pmatrix}2\\6\\3\end{pmatrix}=\begin{pmatrix}6\\12\\0\end{pmatrix}-\frac{12}{7}\begin{pmatrix}2\\6\\3\end{pmatrix}$$

$$=\frac{6}{7}\begin{pmatrix}3\\2\\-6\end{pmatrix},$$

$$\mathbf{N}(1)=\frac{1}{7}\begin{pmatrix}3\\2\\-6\end{pmatrix},\qquad\boxed{2}$$

$$\kappa(1)=\frac{|\mathbf{T}'(1)|}{|\gamma'(1)|}=\frac{|\lambda\,\mathbf{N}(1)|}{|\gamma'(1)|^2}=\frac{6}{49}.$$

Alternatively, compute the curvature as

$$\kappa(1)=\frac{|\gamma'(1)\times\gamma''(1)|}{|\gamma'(1)|^3}=\frac{1}{7^3}\left|\begin{pmatrix}2\\6\\3\end{pmatrix}\times\begin{pmatrix}6\\12\\0\end{pmatrix}\right|=\frac{1}{7^3}\left|\begin{pmatrix}-36\\18\\-12\end{pmatrix}\right|=\frac{6}{7^3}\left|\begin{pmatrix}-6\\3\\-2\end{pmatrix}\right|=\frac{6}{7^2}.$$

$\Longrightarrow$ The radius of the osculating circle of C in $(0,3,2)$ is

$$\frac{1}{\kappa(1)}=\frac{49}{6},\qquad\boxed{1}$$

and the center is

$$\gamma(1)+\frac{1}{\kappa(1)}\,\mathbf{N}(1)=\begin{pmatrix}0\\3\\2\end{pmatrix}+\frac{7}{6}\begin{pmatrix}3\\2\\-6\end{pmatrix}=\frac{1}{6}\begin{pmatrix}21\\32\\-30\end{pmatrix}=\begin{pmatrix}7/2\\16/3\\-5\end{pmatrix}.\qquad\boxed{1}$$

Remarks:

$$\sum_{3}=5$$

$$\sum_{\text{Midterm 2}}=20+2$$
