Math 241 Calculus III (Prof. Honold) **Midterm Exam 1** 2023/03/13

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The area of the parallelogram spanned by $(1,a,2)$ and $(a,1,2)$ is equal to 9 for

   ■ $a=-2$ ☐ $a=-1$ ☐ $a=0$ ☐ $a=1$ ☐ $a=2$

2. The distance from the point $(1,2,6)$ to the plane spanned by $(2,1,0)$, $(0,2,1)$, $(1,0,2)$ is equal to

   ☐ $\sqrt{2}$ ☐ $\sqrt{3}$ ■ $2\sqrt{3}$ ☐ 6 ☐ $3\sqrt{2}$

3. The distance from the point $(2,2,6)$ to the line $2x+y=2y+z=1$ is equal to

   ☐ $\frac{1}{3}\sqrt{6}$ ☐ $\sqrt{6}$ ■ $\frac{5}{3}\sqrt{6}$ ☐ $\frac{7}{3}\sqrt{6}$ ☐ $3\sqrt{6}$

4. $\mathbf{A}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$ satisfy $\mathbf{Av}=-\mathbf{v}$ if

   ☐ $\phi=22.5^\circ$ ☐ $\phi=45^\circ$ ☐ $\phi=67.5^\circ$ ☐ $\phi=90^\circ$ ■ $\phi=112.5^\circ$

5. The smallest distance d^* from the curve $f(t)=(1,0,0)+t(1,-1,0)+t^2(0,1,-1)$, $t\in\mathbb{R}$ to the origin satisfies

   ☐ $d^*=0$ ☐ $d^*\in\left(0,\frac{1}{2}\right)$ ☐ $d^*=\frac{1}{2}$ ■ $d^*\in\left(\frac{1}{2},1\right)$ ☐ $d^*=1$

6. The maximum curvature of the curve $f(t)$ in Question 5 is

   ■ $\frac{4}{3}\sqrt{2}$ ☐ $16\sqrt{3}$ ☐ $2\sqrt{6}$ ☐ $\frac{3}{4}\sqrt{3}$ ☐ $8\sqrt{3}$

7. The tangent to the curve $g(t)=(t,t^2,t^4)$, $t\in\mathbb{R}$ in the point $(1,1,1)$ intersects the plane $ax+y-2z=2023$ unless

   ☐ $a=10$ ☐ $a=1$ ■ $a=6$ ☐ $a=3$ ☐ $a=0$

8. For the twisted cubic $f(t)=(t,t^2,t^3)$, $t\in\mathbb{R}$ the unit normal vector $\mathbf{N}(1)$ is a positive multiple of

   ☐ $(-11,8,9)$ ☐ $(11,8,-9)$ ☐ $(11,-8,9)$ ☐ $(-11,8,-9)$
   ■ $(-11,-8,9)$

9. The arc length of the curve $g(t)=\big(3t\sin(2t),4t^{3/2},3t\cos(2t)\big)$, $t\in[0,5]$ is

   ☐ 30 ☐ 45 ☐ 60 ☐ 75 ■ 90

10. For a differentiable curve $\gamma=\gamma(t)$ in $\mathbb{R}^3$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left(|\gamma|^2\gamma\right)$ is equal to

    ☐ $2|\gamma|\,|\gamma'|\,\gamma+|\gamma|^2\gamma'$ ☐ $2\gamma+|\gamma|^2\gamma'$ ■ $2(\gamma\cdot\gamma')\gamma+(\gamma\cdot\gamma)\gamma'$
    ☐ $2|\gamma|\,\gamma+|\gamma|^2\gamma'$ ☐ $2|\gamma|\,\gamma'+(\gamma\cdot\gamma)\gamma'$

Time allowed: 45 min CLOSED BOOK ***Good luck!***

Notes

1 We have

$$\begin{pmatrix}1\\a\\2\end{pmatrix}\times\begin{pmatrix}a\\1\\2\end{pmatrix}=\begin{pmatrix}2a-2\\2a-2\\1-a^2\end{pmatrix},$$

$$\left|\begin{pmatrix}1\\a\\2\end{pmatrix}\times\begin{pmatrix}a\\1\\2\end{pmatrix}\right|^2=2(4a^2-8a+4)+1-2a^2+a^4=a^4+6a^2-16a+9.$$

This is equal to $9^2=81$ for $a=-2$ and no other value of a in the offered range.

2 The plane has equation $x+y+z=3$ and normal vector $\mathbf{n}=(1,1,1)$. The distance is equal to the length of the orthogonal projection of $\mathbf{b}-\mathbf{a}=(1,2,6)-(2,1,0)=(-1,1,6)$ onto $\mathbb{R}\mathbf{n}$, which is

$$\frac{(-1,1,6)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}\begin{pmatrix}1\\1\\1\end{pmatrix}=2\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

3 Setting $x=t$ gives $y=1-2t$, $z=1-2y=1-2(1-2t)=-1+4t$, so that the line is $\{(t,1-2t,-1+4t);t\in\mathbb{R}\}=(0,1,-1)+\mathbb{R}(1,-2,4)$. In order to obtain the distance d, we orthogonally project $\mathbf{b}-\mathbf{a}=(2,2,6)-(0,1,-1)=(2,1,7)$ onto $\mathbb{R}(1,-2,4)$, subtract the resulting vector from $(2,1,7)$ and take the length:

$$\begin{aligned}d&=\left|\begin{pmatrix}2\\1\\7\end{pmatrix}-\frac{(2,1,7)\cdot(1,-2,4)}{(1,-2,4)\cdot(1,-2,4)}\begin{pmatrix}1\\-2\\4\end{pmatrix}\right|=\left|\begin{pmatrix}2\\1\\7\end{pmatrix}-\frac{28}{21}\begin{pmatrix}1\\-2\\4\end{pmatrix}\right|=\left|\begin{pmatrix}2\\1\\7\end{pmatrix}-\frac{4}{3}\begin{pmatrix}1\\-2\\4\end{pmatrix}\right|\\&=\left|\frac{1}{3}\begin{pmatrix}6-4\\3+8\\21-16\end{pmatrix}\right|=\frac{1}{3}\left|\begin{pmatrix}2\\11\\5\end{pmatrix}\right|=\frac{1}{3}\sqrt{150}=\frac{5}{3}\sqrt{6}.\end{aligned}$$

4 The matrix $\mathbf{A}$ is the reflection matrix $S(45^\circ)$, whose axis in polar coordinates is given by $\phi=22.5^\circ$. Vectors are mapped to their negatives if they are on the line through the origin orthogonal to the axis, i.e., if $\phi=22.5^\circ+90^\circ=112.5^\circ$.

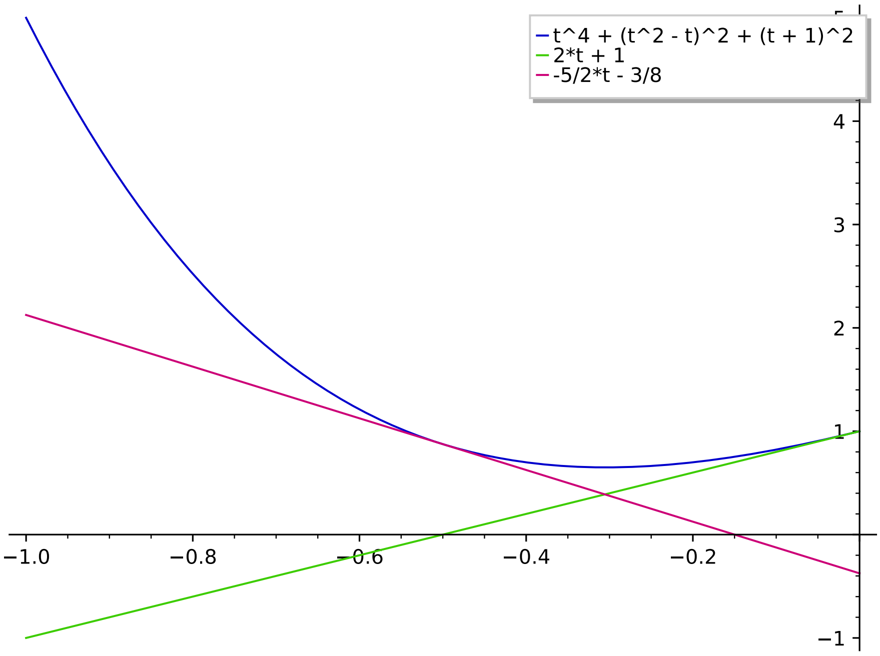
5 The easiest solution is to observe that the given curve, $f(t)=(1+t,-t+t^2,-t^2)$, is contained in the plane $x+y+z=1$, which has distance $1/\sqrt{3}\approx0.577$ from the origin, and contains a point at distance less than 1 from the origin, viz. $f\left(-\frac{1}{2}\right)=\left(\frac{1}{2},\frac{3}{4},-\frac{1}{4}\right)$, which has squared length $\frac{7}{8}$. Thus necessarily $d^*\in\left(\frac{1}{2},1\right)$.

Another proof of $d^*>\frac{1}{2}$ uses $1+t>\frac{1}{2}$ for $t>-\frac{1}{2}$, $-t+t^2\geq\frac{1}{2}+\frac{1}{4}>\frac{1}{2}$ for $t\leq-\frac{1}{2}$, so that for any t at least one coordinate of $f(t)$ is $>\frac{1}{2}$.

One can also solve the problem with the Calculus I machinery for determining extrema of one-variable functions, but this is more involved because of lack of a calculator. We have

$$\begin{aligned}g(t)&:=|f(t)|^2=(1+t)^2+(-t+t^2)^2+(-t^2)^2=2t^4-2t^3+2t^2+2t+1,\\g'(t)&=8t^3-6t^2+4t+2,\\g''(t)&=24t^2-12t+4=4(6t^2-3t+1).\end{aligned}$$

Since g'' is positive everywhere, g' is strictly increasing and has a unique zero t^*, which must be the unique minimum of the (convex) function g. Since $g'(0)=2>0$, $g'(-1)=-16<0$, we must have $t^*\in(-1,0)$. The graph of g must be above the tangents at $t=0$ and $t=-1$, viz. $y=1+2t$ and $y=5-16(t+1)$, but this is not sufficient to bound $g(t)$ nontrivially from below. Refining the argument, one finds $g'\left(-\frac{1}{2}\right)=-5/2<0$, from which $t^*\in\left(-\frac{1}{2},0\right)$. Using $y=1+2t$ and the tangent at $t=-\frac{1}{2}$, viz. $y=\frac{7}{8}-\frac{5}{2}\left(t+\frac{1}{2}\right)$, as lower bounds gives $g(t)\geq\frac{7}{18}>\frac{1}{4}$; see picture. This is sufficient for the estimate $d^*>\frac{1}{2}$.



A numerical computation yields $t^* \approx -0.3045$, $g(t^*) = |f(t^*)|^2 \approx 0.6501$, $|f(t^*)| \approx 0.8063$.

6 We have

$$\begin{aligned}
f(t) &= (1+t, -t+t^2, -t^2), \\
f'(t) &= (1, -1+2t, -2t), \\
f''(t) &= (0, 2, -2), \\
f'(t) \times f''(t) &= (2, 2, 2), \\
|f'(t)| &= \sqrt{1+(2t-1)^2+4t^2} = \sqrt{8t^2-4t+2}, \\
\kappa(t) &= \frac{|f'(t) \times f''(t)|}{|f'(t)|^3} = \frac{2\sqrt{3}}{(8t^2-4t+2)^{3/2}}.
\end{aligned}$$

The minimum of the quadratic in the denominator is at $t = 1/4$ (use “completing the square” or set the derivative of the quadratic equal to zero), and therefore the maximum curvature is $\kappa(1/4) = \frac{2\sqrt{3}}{(3/2)^{3/2}} = \frac{2\sqrt{3}\cdot 2\sqrt{2}}{3\sqrt{3}} = \frac{4}{3}\sqrt{2}$.

7 We have

$$\begin{aligned}
g'(t) &= (1, 2t, 4t^3), \\
T &= g(1) + \mathbb{R}g'(1) = (1,1,1) + \mathbb{R}(1,2,4).
\end{aligned}$$

T will intersect the plane unless it is parallel to it, which is equivalent to $(1,2,4) \perp (a,1,-2)$, i.e., $(1,2,4)\cdot(a,1,-2) = a+2-8 = a-6 = 0$.

8 We have

$$\begin{aligned}
f'(t) &= (1, 2t, 3t^2), \\
f''(t) &= (0, 2, 6t), \\
f''(1) - \frac{f''(1)\cdot f'(1)}{f'(1)\cdot f'(1)} f'(1) &= (0,2,6) - \frac{(0,2,6)\cdot(1,2,3)}{(1,2,3)\cdot(1,2,3)}(1,2,3) \\
&= \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} - \frac{22}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -11/7 \\ -8/7 \\ 9/7 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -11 \\ -8 \\ 9 \end{pmatrix}.
\end{aligned}$$

9 We have

$$\begin{aligned}
g'(t) &= 3\sin(2t) + 6t\cos(2t) + 6t^{1/2} + 3\cos(2t) - 6t\sin(2t),\\
\left|g'(t)\right|^2 &= 9 + 36t + 36t^2,\\
\mathrm{L}(g) &= \int_0^5 \sqrt{9+36t+36t^2}\,\mathrm{d}t = 3\int_0^5 \sqrt{1+4t+4t^2}\,\mathrm{d}t = 3\int_0^4 1+2t\,\mathrm{d}t\\
&= 3\left[t+t^2\right]_0^5 = 90.
\end{aligned}$$

10 $\frac{\mathrm{d}}{\mathrm{d}t}\left(|\gamma|^2\gamma\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left((\gamma\cdot\gamma)\gamma\right) = (\gamma'\cdot\gamma)\gamma + (\gamma\cdot\gamma')\gamma + (\gamma\cdot\gamma)\gamma' = 2(\gamma\cdot\gamma')\gamma + (\gamma\cdot\gamma)\gamma'$