16.5 Curl and Divergence

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Curl

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

For memorization, we can rewrite it with gradient:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \text{curl } \mathbf{F}\end{aligned}$$

Hence,

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

Theorem

If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl } \nabla f = \mathbf{0}$$

Theorem

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl } \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Divergence

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$ and $\frac{\partial R}{\partial z}$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

For memorization, we can rewrite it with gradient:

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Proof:

$$\begin{aligned}
\text{div curl } \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\
&= 0
\end{aligned}$$