14.7 Maximum and Minimum Values

Local Maximum and Minimum Values

Definition

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ has a local minimum at (a, b) and $f(a, b)$ is a local minimum value.

Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
Proof:
Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem. But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$. Similarly, $f_y(a, b) = 0$.

Critical Points

A point (a, b) is called a critical point (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or if one or both partial derivatives do not exist.
Theorem says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in single-variable calculus, not all critical points give rise to maxima or minima.

Second Derivative Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suupose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so that (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum of f.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum of f.
(c) If $D < 0$, then (a, b) is a saddle point of f.
Proof:
(a) We compute the second-order directional derivative of f in the direction of $\mathbf{u} = \langle h, k \rangle$. The first-order derivative is given by

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = f_x h + f_y k$$

Applying the directional derivative formula again, we obtain

$$\begin{aligned} D_{\mathbf{u}}^2 f &= \nabla(D_{\mathbf{u}} f) \cdot \mathbf{u} \\ &= (f_{xx} h + f_{xy} k) h + (f_{yx} h + f_{yy} k) k \\ &= f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 \end{aligned}$$

If we complete the square in this expression, we obtain

$$D_{\mathbf{u}}^2 f = f_{xx}(h + \frac{f_{xy}}{f_{xx}} k)^2 + \frac{k^2}{f_{xx}}(f_{yy} f_{xx} - f_{xy}^2)$$

We are given that $f_{xx}(a, b) > 0$ and $D(a, b) > 0$.