14.6 Directional Derivatives and the Gradient Vector

Directional Derivatives

Definition

The Directional Derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$, and

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Proof:
Let $g(h) = f(x_0 + ha, y_0 + hb) = f(x, y)$ where $x = x_0 + ha$ and $y = y_0 + hb$
Then

$$\begin{aligned} g'(0) &= \lim_{h \to 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}} f(x_0, y_0) \end{aligned}$$

Then we apply the Chain Rule (Case 1) to g to obtain

$$\begin{aligned} g'(h) &= \frac{\partial g}{\partial x}\frac{dx}{dh} + \frac{\partial g}{\partial y}\frac{dy}{dh} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} \\ &= f_x(x, y)a + f_y(x, y)b \end{aligned}$$

Hence

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Thus

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Others

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$. Then

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$$

The Gradient Vector

Definition

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The Gradient Vector and Directional Derivatives

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

Functions of Three Variables

Directional Derivatives

The **Directional Derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

If we use vector notation, we can write the definition of the directional derivative in the compact form

$$D_{\mathbf{u}}f(\mathbf{x_0}) = \lim_{h \to 0} \frac{f(\mathbf{x_0} + h\mathbf{u}) - f(\mathbf{x_0})}{h}$$

where $\mathbf{x_0} = \langle x_0, y_0, z_0\rangle$ and $\mathbf{u} = \langle a, b, c\rangle$.

$$D_{\mathbf{u}}f(\mathbf{x_0}) = \nabla f(\mathbf{x_0}) \cdot \mathbf{u}$$

Maximizing the Directional Derivative

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$. Let t be the parameter. Then $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.

$$F(x(t), y(t), z(t)) = k$$

Hence

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

Since $\nabla F = \langle F_x, F_y, F_z\rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$, we can write this equation in the form

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

In particular, when $t = t_0$, we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$, so

$$\nabla F(\mathbf{x_0}) \cdot \mathbf{r}'(t_0) = 0$$

That means the *gradient vector* at P, $\nabla F(\mathbf{x_0},)$, is perpendicular to *tangent vector $\mathbf{r}'(t_0)$ to any curve C on the* *surface S that passes through P*. If $\nabla F(\mathbf{x_0}) \neq \mathbf{0}$, it's natural to defind the **tanget plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ that passes through P and has normal vector $\nabla F(\mathbf{x_0})$. That is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

And the symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Properties of the Gradient Vector

Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x_0}) \neq \mathbf{0}$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of increase is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.