14.4 Tangent Planes and Linear Approximations

Tangent Planes

Equation of a Tangent Plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear Approximations

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linearization of f at (a, b).
And the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation of f at (a, b).

Differentials

Differentiability

Definition

If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as Δx and $\Delta y \to 0$.

Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Differentials

Definition

For a differentiable function of two variables $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values. Then thee differential dz, also called the total differential, is defined by

$$dz = f_x(a, b)dx + f_y(a, b)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Differential and Linear Approximation

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation of f at (a, b) is

$$f(x, y) \approx f(a, b) + dz$$

Functions of Three or More Variables

Similar to the two-variable case.