Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The surface in $\mathbb{R}^3$ with equation $x^4 + y^4 + z^4 + 4xyz + 1 = 0$ is smooth.

b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(x,y) \to +\infty$ for $|(x,y)| \to \infty$. Then f has a global minimum.

c) There exists a differentiable function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with at least 2023 strict local maxima.

d) A function $f(x,y)$ is integrable over $\Delta = \{(x,y) \in \mathbb{R}^2; x \geq 0, y \geq 0, x+y \leq 1\}$ if and only if $(u,v) \mapsto f\big(u(1-v), uv\big) \cdot u$ is integrable over the unit square $[0,1]^2$.

e) There exists a continuous vector field in $\mathbb{R}^2$ that is conservative and at every point of the unit circle nonzero and tangent to the unit circle.

f) Integrals of the differential 1-form $\frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2}$ in the region $\{(x,y) \in \mathbb{R}^2; x > 0 \vee y > 0\}$ are independent of path.

Question 2 (ca. 11 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = (1 + 6\,xy)\mathrm{e}^{-x^2-y^2}.$$

a) Determine all critical points $\mathbf{p}$ of f, their types using the 2nd-order partial derivatives test (Hesse matrix test), and the corresponding values $f(\mathbf{p})$.

Hint: There are 5 critical points. When computing 2nd-order derivatives it is best to use the identity $f_x(x,y) = 6y\,\mathrm{e}^{-x^2-y^2} - 2x\,f(x,y)$, and similarly for $f_y(x,y)$.

b) Does f have global extrema?

c) For $A = \{(x,y) \in \mathbb{R}^2; x > 0, y > 0\}$ compute the Lebesgue integral $\int_A f(x,y)\,\mathrm{d}^2(x,y)$.

Question 3 (ca. 7 marks)

Using the method of Lagrange multipliers, solve the optimization problem

$$\begin{aligned} &\text{Maximize} && z \\ &\text{subject to} && x^2 + y^2 + z^2 = x^3 + y^3 = 9. \end{aligned}$$

Note: Required are (i) a proof that the optimization problem has a solution, (ii) the optimal objective value z^*, and (iii) all optimal solutions (x^*, y^*, z^*).

Question 4 (ca. 5 marks)

For $a > -1$ evaluate

$$\int_0^1 x^a (\ln x)^3\,\mathrm{d}x\,.$$

Hint: Start with $F(a) = \int_0^1 x^a\,\mathrm{d}x$, which is easy to evaluate directly, and show—carefully justifying each step—that F can be differentiated thrice under the integral sign.

Question 5 (ca. 7 marks)

Let K be the solid in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$x \geq 0, \quad x^2 + y^2 \leq 4, \quad 0 \leq z \leq x^2 - y^2.$$

Find the volume of K and the surface area of ∂K.

Solutions

1 a) False. The surface is the (-1)-level set of $g(x,y,z)=x^4+y^4+z^4+4xyz$, which has gradient $\nabla g(x,y,z)=(4x^3+4yz,4y^3+4xz,4z^3+4xy)$. At $(-1,-1,-1)$ we have $\nabla g=0$; since $g(-1,-1,-1)=-1$, this point is on the surface and hence a singular point. $\boxed{2}$

b) True. Since $\lim_{|(x,y)|\to\infty} f(x,y)=+\infty$, there exists $R>0$ such that $f(x,y)>f(0,0)$ for $|(x,y)|>R$. On the compact disk $\overline{\mathrm{B}_R(0,0)}$ te continuous function f attains a minimum m, which is $\leq f(0,0)$. Since $f(x,y)>f(0,0)\geq m$ for all points (x,y) outside the disk, this minimum is a global minimum.

c) True. For $g(x,y)=\sin x+\sin y$ we have

$$\begin{aligned}\nabla g(x,y)&=(\cos x,\cos y)^{\mathsf{T}},\\ \mathbf{H}_g(x,y)&=\begin{pmatrix}-\sin x&0\\0&-\sin y\end{pmatrix}.\end{aligned}$$

All points $\mathbf{p}_{kl}=\left(\frac{(2k+1)\pi}{2},\frac{(2l+1)\pi}{2}\right)$, $k,l\in\mathbb{Z}$, are critical; those with k,l even have $\mathbf{H}_g(\mathbf{p}_{kl})=\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$, and hence provide strict local maxima. $\boxed{2}$

d) True. This is an instance of the change-of-variables theorem. In order to justify this, observe that $T(u,v)=\big(u(1-v),uv\big)$ has $\det\mathbf{J}_T(u,v)=\left|\begin{smallmatrix}1-v&-u\\v&u\end{smallmatrix}\right|=(1-v)u-v(-u)=u>0$ on $(0,1)^2$ and maps $(0,1)^2$ bijectively to Δ°. In order to see the latter, let $x=u(1-v)$, $y=uv$. If $(u,v)\in(0,1)^2$ then $x>0$, $y>0$, and $x+y=u<1$. Conversely, if $(x,y)\in\Delta$ then $u=x+y\in(0,1)$ and $v=\frac{y}{x+y}\in(0,1)$. $\boxed{2}$

e) False. On the unit circle the normalization $F/|F|$ of such a vector field, call it F, is continuous and tangent to the unit circle as well. Because of continuity the tangent direction cannot make a U-turn, and hence either $F(\cos t,\sin t)$ is a positive multiple of $(-\sin t,\cos t)$ for all $t\in[0,2\pi]$, or a negative multiple of $(-\sin t,\cos t)$ for all $t\in[0,2\pi]$. In the first case the line integral along the unit circle in its usual parametrization is positive, in the second case negative. This contradicts conservativity/exactness of the vector field/differential 1-form, which requires line integrals along closed curves to be zero.

f) True. The form, call it ω, is the sum of $\frac{x\,\mathrm{d}x+y\,\mathrm{d}y}{x^2+y^2}$, which is exact (an antiderivative is $\frac{1}{2}\ln(x^2+y^2)$), and the winding form. Since the winding form is locally exact, so is ω. Since $\{(x,y)\in\mathbb{R}^2;x>0\vee y>0\}$ is star shaped with centre, e.g., $(1,1)$, ω is exact (Poincaré's Lemma), and hence $\int_\gamma\omega$ is independent of path. $\boxed{2}$

Remarks: No marks were assigned for wrong answers and answers without justification.

a) Shockingly, almost half of the students falsely claimed that the surface is smooth. In all the surface has 4 singular points, viz. $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(-1,-1,-1)$. Since the suface is invariant under simultaneous sign changes of any two of the three variables x,y,z, it is clear that the first three points must also be singular, and it is not difficult to see that there are no further singular points.

b) Full marks were only assigned for a rigorous proof, which must essentially be the one given above. If students worked with the disk $\mathrm{B}_R(0,0)$ (which is open) instead of $\overline{\mathrm{B}_R(0,0)}$ and mentioned "closed", I haven't subtracted any marks. Many students provided invalid proofs, rephrasing just the statement to prove or relying on "this thing is obvious" and the like. Also it is false to assume that there exists a disk $\overline{\mathrm{B}_R(0,0)}$ such that $f(x_2,y_2) \geq f(x_1,y_1)$ for all points (x_2,y_2) outside the disk and all points (x_1,y_1) inside the disk.

c) For (correct) examples of such functions without any justification I have assigned only 0.5 marks. Many students used related examples such as $\sin x + \cos y$, in which case the maxima are located at points different from those above.

Several students used the example $f(x,y) = (x+y-1)(y-x)(y-2x)\cdots$ from one of our sample exams, but the argument presented there cannot work, since $g(x,y) = (x+y-1)(y-mx)$ has only a saddle point and no other extrema.

Other students used functions like $\sin(x+y)$, $\sin(xy)$, $y\sin x$, which don't have strict local maxima.

Two students discovered a nice example but didn't realize how easy the solution is: Use the function $g(x,y) = \sin^2 x + \sin^2 y$. Without any computation necessary, the maxima of this function are obviously at points (x,y) where $\sin x = \pm 1$ and $\sin y = \pm 1$, which are $\pi/2 \cdot (k,l)$ with k,l odd integers. Since these points form a discrete set, the maxima are strict, and there are of course infinitely many such points.

Another way to construct examples is to periodically repeat a function h with a single maximum. But in order for this to work, the supports of the translates of h should be disjoint. Essentially what one needs is that h has a strict maximum at $(0,0)$ and vanishes outside the unit disk. Such a function can be constructed by rotating a corresponding 1-variable function, but the differentiability requirement makes the solution nontrivial. (In Math285 we will see that even C^∞-functions with these properties exist.) For those answers, which never went into an actual construction, I have assigned 0.5 marks.

d) Many students recognized this as an instance of change-of-variables but were unable to fill in all details. A proof that T is bijective, which requires solving $x = u(1-v) \wedge y = uv$ for u,v was only given by few students. Also many students didn't notice that $\det \mathbf{J}_T(u,v)$ is involved and has to be computed.

e) Only few students noticed that a key property to be exploited is that the line integral of F along the unit circle must be zero. This was honored by 1 mark. Noone noticed that the second key property is that F is nonzero everywhere on the unit circle. (There exist conservative vector fields which are nonzero and tangent to the unit circle in all but two points. Think of related examples from Physics.)

f) Most students showed that the given 1-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$ is locally exact, which requires to to show $P_y = Q_x$. This was worth 1 mark. But many forgot that the shape of the region also plays a role in exactness. The 2nd mark was only assigned when it was mentioned that the region is star-shaped or simply connected.

$$\sum_1 = 12$$

2 a) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned}
f &= (1+6xy)\mathrm{e}^{-x^2-y^2}\\
f_x &= \big(6y-2x(1+6xy)\big)\mathrm{e}^{-x^2-y^2} = (6y-2x-12x^2y)\mathrm{e}^{-x^2-y^2}\\
f_y &= \big(6x-2y(1+6xy)\big)\mathrm{e}^{-x^2-y^2} = (6x-2y-12xy^2)\mathrm{e}^{-x^2-y^2}\\
yf_x - xf_y &= (6y^2-6x^2)\mathrm{e}^{-x^2-y^2}.
\end{aligned}$$

Hence $\nabla f(x,y) = (0,0)$ implies $x = \pm y$.

Clearly $\mathbf{p}_0 = (0,0)$ is critical, and $f(\mathbf{p}_0) = 1$. $\boxed{1}$

Assuming $(x,y) \neq (0,0)$, we distinguish two mutually exclusive cases:

<u>Case 1: $x = y$</u> Here $f_x = 0$ gives $4x - 12x^3 = 4x(1-3x^2) = 0$, $x = \pm\frac{1}{3}\sqrt{3}$. This yields the two critical points $\mathbf{p}_1 = \big(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\big)$, $\mathbf{p}_2 = \big(-\frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\big)$. The corresponding values are $f(\mathbf{p}_1) = f(\mathbf{p}_2) = 3\,\mathrm{e}^{-2/3}$. $\boxed{2}$

<u>Case 2: $x = -y$</u> Here $f_x = 0$ gives $-8x+12x^3 = 4x(-2+3x^2) = 0$, $x = \pm\frac{\sqrt{2}}{\sqrt{3}} = \pm\frac{1}{3}\sqrt{6}$. This yields the two critical points $\mathbf{p}_3 = \big(\frac{1}{3}\sqrt{6}, -\frac{1}{3}\sqrt{6}\big)$, $\mathbf{p}_4 = \big(-\frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{6}\big)$. The corresponding values are $f(\mathbf{p}_3) = f(\mathbf{p}_4) = -3\,\mathrm{e}^{-4/3}$. $\boxed{2}$

For the determination of the types we compute the 2nd-order derivatives using the hint:

$$\begin{aligned}
f_x &= 6y\,\mathrm{e}^{-x^2-y^2} - 2xf,\\
f_y &= 6x\,\mathrm{e}^{-x^2-y^2} - 2yf,\\
f_{xx} &= -12\,xy\,\mathrm{e}^{-x^2-y^2} - 2f - 2xf_x,\\
f_{xy} &= f_{yx} = 6(1-2xy)\mathrm{e}^{-x^2-y^2} - 2xf_y,\\
f_{yy} &= -12\,xy - 2f - 2yf_y.
\end{aligned}$$

Since $f_x = f_y = 0$ at a critical point $\mathbf{p}$ and $f(\mathbf{p})$ has already been determined, this makes it easy to compute the corresponding Hesse matrices.

$$\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} -2 & 6\\ 6 & -2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \mathbf{H}_f(\mathbf{p}_2) = \mathrm{e}^{-2/3}\begin{pmatrix} -10 & 2\\ 2 & -10 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_3) = \mathbf{H}_f(\mathbf{p}_4) = \frac{1}{3}\,\mathrm{e}^{-2/3}\begin{pmatrix} 26 & -2\\ -2 & 26 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_0)$ is indefinite (determinant < 0), the point $\mathbf{p}_0$ is a saddle point. $\boxed{1}$

Since $\mathbf{H}_f(\mathbf{p}_1) = \mathbf{H}_f(\mathbf{p}_2)$ is negative definite (determinant > 0, top-left entry < 0), the points $\mathbf{p}_1$, $\mathbf{p}_2$ are strict local maxima. $\boxed{1}$

Since $\mathbf{H}_f(\mathbf{p}_3) = \mathbf{H}_f(\mathbf{p}_4)$ is positive definite (determinant > 0, top-left entry > 0), the points $\mathbf{p}_3$, $\mathbf{p}_4$ are strict local minima. $\boxed{1}$

b) Yes. The points $\mathbf{p}_1$, $\mathbf{p}_2$ are the global maxima (with value $f(\mathbf{p}_1)=f(\mathbf{p}_2)=3\,\mathrm{e}^{-2/3}$), and $\mathbf{p}_3$, $\mathbf{p}_4$ are the global minima (with value $f(\mathbf{p}_1)=f(\mathbf{p}_2)=-3\,\mathrm{e}^{-4/3}$).

The existence of a global maximum follows from from $\lim_{|(x,y)|\to\infty} f(x,y)=0$ (clear from the exponential decrease of $(x,y)\mapsto \mathrm{e}^{-x^2-y^2}$) and the fact that f attains a positive value. Similarly, the existence of a global minimum follows from from $\lim_{|(x,y)|\to\infty} f(x,y)=0$ and the fact that f attains a negative value. $\boxed{1}$

Since the global maxima/minima must be among the local maxima/minima, it is then easy to identify them.

c) We have

$$\begin{aligned}\int_A f(x,y)\mathrm{d}^2(x,y) &= \int_A \mathrm{e}^{-x^2-y^2}\mathrm{d}^2(x,y)+6\int_A xy\,\mathrm{e}^{-x^2-y^2}\mathrm{d}^2(x,y)\\ &=\frac{1}{4}\int_{\mathbb{R}^2}\mathrm{e}^{-x^2-y^2}\mathrm{d}^2(x,y)+6\int_0^\infty x\,\mathrm{e}^{-x^2}\,\mathrm{d}x\int_0^\infty y\,\mathrm{e}^{-y^2}\,\mathrm{d}y\\ &=\pi/4+6\left(\left[-\frac{1}{2}\,\mathrm{e}^{-x^2}\right]_0^\infty\right)^2\\ &=\frac{\pi}{4}+6\left(\frac{1}{2}\right)^2=\frac{\pi+6}{4}.\end{aligned} \qquad \boxed{2}$$

Remarks:

$$\sum_2 = 11$$

3 The continuous function $f(x,y,z)=z$ attains a maximum on the set $S=\{(x,y,z)\in\mathbb{R}^3; x^2+y^2+z^2=9, x^3+y^3=9\}$, which is closed (because it is defined by equality constraints involving continuous functions) and bounded (because it is contained in a sphere). This shows that the optimization problem has at least one solution. $\boxed{1}$

Setting $\mathbf{g}=(g_1,g_2)$ with $g_1(x,y,z)=x^2+y^2+z^2$, $g_2(x,y,z)=x^3+y^3$, the task is to minimize f on $\mathbb{R}^3$ under the constraint $\mathbf{g}(x,y,z)=(9,9)$.

$$\nabla f(x,y,z)=(0,0,1),\qquad \mathbf{J}_\mathbf{g}(x,y,z)=\begin{pmatrix}2x & 2y & 2z\\ 3x^2 & 3y^2 & 0\end{pmatrix}.$$

The Lagrange Multiplier Theorem is applicable in points $(x,y,z)\in S$ for which $\mathbf{J}_\mathbf{g}(x,y,z)$ has rank 2. The 2×2 subdeterminants of $\mathbf{J}_\mathbf{g}(x,y,z)$ are $6xy^2-6x^2y=6xy(y-x)$, $-6x^2z$, $-6y^2z$. If $\operatorname{rank}\mathbf{J}_\mathbf{g}(x,y,z)\leq 1$ then all three subdeterminants are zero, which implies either $z=0\wedge(x=0\vee y=0\vee x=y)$ or $x=y=0$. In the first case $z=0$ gives $x^2+y^2=x^3+y^3=9$, and each of the possibilities $x=0$, $y=0$, $x=y$ produces a contradiction. In the second case $x^3+y^3=0$, contradicting $x^3+y^3=9$. Thus $\operatorname{rank}\mathbf{J}_\mathbf{g}=2$ for all points on S. $\boxed{1\tfrac{1}{2}}$

Hence every optimal solution must satisfy $\nabla f(x,y,z)=\lambda\nabla g_1(x,y,z)+\mu\nabla g_2(x,y,z)$

for some $\lambda, \mu \in \mathbb{R}$, giving the system of equations

$$\begin{aligned} 0 &= \lambda\, 2x + \mu\, 3x^2, \\ 0 &= \lambda\, 2y + \mu\, 3y^2, \\ 1 &= \lambda\, 2z, \\ x^2 + y^2 + z^2 &= 9, \\ x^3 + y^3 &= 9. \end{aligned}$$ $\boxed{2\frac{1}{2}}$

The first two equations represent the linear system

$$\begin{pmatrix} 2x & 3x^2 \\ 2y & 3y^2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\lambda = 0$ is impossible in view of the 3rd equation, a solution must have $\begin{vmatrix} 2x & 3x^2 \\ 2y & 3y^2 \end{vmatrix} = 0$, which implies $x = 0 \lor y = 0 \lor x = y$. This gives 6 candidate points, viz.,

$$\left(0, 9^{1/3}, \pm\left(9 - 9^{2/3}\right)^{1/2}\right), \quad \left(9^{1/3}, 0, \pm\left(9 - 9^{2/3}\right)^{1/2}\right), \quad \left((9/2)^{1/3}, (9/2)^{1/3}, \pm\left(\left(9 - 2(9/2)^{2/3}\right)\right)^{1/2}\right).$$ $\boxed{3}$

Since $2(9/2)^{2/3} = 2^{1/3}9^{2/3} > 9^{2/3}$, we have $\left(\left(9 - 2(9/2)^{2/3}\right)\right)^{1/2} < \left(9 - 9^{2/3}\right)^{1/2}$, and hence the points on S maximizing z are

$$\left(0, 9^{1/3}, \left(9 - 9^{2/3}\right)^{1/2}\right), \quad \left(9^{1/3}, 0, \left(9 - 9^{2/3}\right)^{1/2}\right).$$ $\boxed{1}$

Remarks:

$$\sum_3 = 9$$

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4 $F(a)$ is defined for $a > -1$ and can be evaluated easily as follows:

$$F(a) = \int_0^1 x^a \,\mathrm{d}x = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1^{a+1}}{a+1} - \frac{0^{a+1}}{a+1} = \frac{1}{a+1}.$$ $\boxed{\frac{1}{2}}$

(Strictly speaking, for $a < 0$ we should first integrate over $[\epsilon, 1]$, $\epsilon > 0$, which gives the value $\frac{1}{a+1} - \frac{\epsilon^{a+1}}{a+1}$, and then let $\epsilon \to 0$.)

Writing $f(a, x) = x^a$, we have $f_a(a, x) = \frac{\partial}{\partial a}\mathrm{e}^{(\ln x)a} = (\ln x)x^a$. In order to justify differentiation under the integral sign, we need for fixed $a_0 \in (-1, +\infty)$ an integrable upper bound $\Phi(x)$ for $|(\ln x)x^a| = -(\ln x)x^a$ on $(0, 1)$ that is independent of a in some neighborhood of a_0. Choosing δ strictly between -1 and a_0, the intervall $(\delta, +\infty)$ is a neighborhood of a_0, and for $a \in (\delta, +\infty)$ we have

$$-(\ln x)x^a \leq -(\ln x)x^\delta,$$

since $x \in (0, 1)$. The function $\Phi_\delta(x) := -(\ln x)x^\delta$, $x \in (0, 1)$, are integrable for $\delta > -1$, because $-\ln x$ for $x \downarrow 0$ grows more slowly than any (small) negative power of x, showing that $-(\ln x)x^\delta \leq x^{\delta'}$ with δ' slightly smaller than δ but still > -1. $\boxed{2}$

Thus differentiation under the integral sign is justified and gives

$$F'(a) = \int_0^1 (\ln x) x^a \,\mathrm{d}x \quad \text{for } a > -1.$$

In the same way one proves

$$F''(a) = \int_0^1 (\ln x)^2 x^a \,\mathrm{d}x, \quad F'''(a) = \int_0^1 (\ln x)^3 x^a \,\mathrm{d}x \quad \text{for } a > -1, \qquad \boxed{1}$$

using that $(-1)^k(\ln x)^k$ for $x \downarrow 0$ for any positive integer k still grows more slowly than any (small) negative power of x.

On the other hand, using $F(a) = \frac{1}{a+1}$ we have

$$\begin{aligned}
\int_0^1 (\ln x) x^a \,\mathrm{d}x &= F'(a) = -\frac{1}{(a+1)^2},\\
\int_0^1 (\ln x)^2 x^a \,\mathrm{d}x &= F''(a) = \frac{2}{(a+1)^3},\\
\int_0^1 (\ln x)^3 x^a \,\mathrm{d}x &= F'''(a) = -\frac{6}{(a+1)^4}.
\end{aligned} \qquad \boxed{1\tfrac{1}{2}}$$

Remarks:

$$\sum_4 = 5$$

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5 a) For $(x,y) \in \mathbb{R}^2$ the corresponding (x,y)-section $K_{x,y} = \{z \in \mathbb{R}; (x,y,z) \in K\}$ is nonempty iff $x^2 - y^2 \geq 0$, which under the assumption $x \geq 0$ simplifies to $-x \leq y \leq x$, and is an interval of length $x^2 - y^2$ in that case. The set $S = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \leq 4, x \geq 0, -x \leq y \leq x\}$ (projection of K onto the (x,y)-plane) is a sector of the disk with radius $r = 2$ centered at the origin and in polar coordinates given by $-\pi/4 \leq \theta \leq \pi/4$.

$$\begin{aligned}
\Longrightarrow \operatorname{vol}_3(K) &= \int_K 1 \,\mathrm{d}^3(x,y,z) = \int_S x^2 - y^2 \,\mathrm{d}^2(x,y)\\
&= \int\limits_{\substack{0 \leq r \leq 2\\ -\pi/4 \leq \theta \leq \pi/4}} \bigl((r\cos\theta)^2 - (r\sin\theta)^2\bigr) r \,\mathrm{d}^2(r,\theta)\\
&= \left(\int_0^2 r^3 \,\mathrm{d}r\right)\left(\int_{-\pi/4}^{\pi/4} \cos^2\theta - \sin^2\theta \,\mathrm{d}\theta\right)\\
&= \left[\frac{r^4}{4}\right]_0^2 \left[\frac{1}{2}\sin(2\theta)\right]_{-\pi/4}^{\pi/4} = 4 \cdot 1 = 4.
\end{aligned} \qquad \boxed{3}$$

b) The smooth part of ∂K consists of 3 parts, viz.,

$$\begin{aligned}
S_1 &= \{(x,y,0); x^2 + y^2 < 4, x > 0, -x < y < x\},\\
S_2 &= \{(x,y,x^2 - y^2); (x,y) \in S^\circ\},\\
S_3 &= \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 = 4, x > 0, -x < y < x, 0 < z < x^2 - y^2\}.
\end{aligned}$$

The remaining parts are smooth curves and don't matter for surface integration.

Since S_1 is S embedded into $\mathbb{R}^3$, the area of S_1 is $2^2\pi/4 = \pi$.

S_2 is the graph of the function $f(x,y) = x^2 - y^2$, $(x,y) \in S^\circ$.

$$\begin{aligned}\implies \operatorname{vol}_2(S_2) &= \int_S \sqrt{1 + |\nabla f(x,y)|^2}\, \mathrm{d}^2(x,y)\\ &= \int_S \sqrt{1+4x^2+4y^2}\, \mathrm{d}^2(x,y) \qquad \text{since } \nabla f(x,y) = (2x,-2y)\\ &= \int\limits_{\substack{0\le r\le 2\\ -\pi/4\le\theta\le\pi/4}} \sqrt{1+4r^2}\, r\, \mathrm{d}^2(r,\theta)\\ &= \frac{\pi}{2}\int_0^2 r\sqrt{1+4r^2}\,\mathrm{d}r\\ &= \frac{\pi}{2}\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^2 = \frac{\pi}{24}\left(17\sqrt{17}-1\right).\end{aligned}$$ 2

A (regular, bijective) parametrization of S_3 is

$$\gamma(\theta,z) = \begin{pmatrix} 2\cos\theta\\ 2\sin\theta\\ z\end{pmatrix}, \quad (\theta,z)\in\Omega$$

with $\Omega = \bigl\{(\theta,z)\in\mathbb{R}^2; -\pi/4<\theta<\pi/4, 0<z<4\cos(2\theta)\bigr\}$ (using $\cos^2\theta - \sin^2\theta = \cos(2\theta)$).

$$\begin{aligned}\mathbf{J}_\gamma(\theta,z) &= \begin{pmatrix} -2\sin\theta & 0\\ 2\cos\theta & 0\\ 0 & 1\end{pmatrix},\\ \mathbf{J}_\gamma(\theta,z)^\mathsf{T}\mathbf{J}_\gamma(\theta,z) &= \begin{pmatrix}4&0\\0&1\end{pmatrix},\\ g^\gamma(\theta,z) &= \sqrt{\det\begin{pmatrix}4&0\\0&1\end{pmatrix}} = 2,\\ \operatorname{vol}_2(S_3) &= \int_\Omega 2\,\mathrm{d}^2(\theta,z)\\ &= 2\int_{-\pi/4}^{\pi/4} 4\cos(2\theta)\mathrm{d}\theta\\ &= 4\left[\sin(2\theta)\right]_{-\pi/4}^{\pi/4} = 4\cdot 2 = 8.\end{aligned}$$ 2

In all we obtain

$$\begin{aligned}\operatorname{vol}_2(S) = \operatorname{vol}_2(S_1)+\operatorname{vol}_2(S_2)+\operatorname{vol}_2(S_3) = \pi + \frac{\pi}{24}\left(17\sqrt{17}-1\right)+8 =\\ = \frac{\pi}{24}\left(17\sqrt{17}+23\right)+8.\end{aligned}$$ 1

Remarks:

$\sum_5 = 8$

$$\sum_{\text{Final Exam}} = 12 + 11 + 9 + 5 + 8 = 45 = 40 + 5$$