Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) For any $A \in \mathbb{R}$ the surface in $\mathbb{R}^3$ with equation $x^3 + y^3 + z^3 + A\,xyz = 1$ is smooth.

b) Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(x,y) \to 0$ for $|(x,y)| \to \infty$. Then f has a global extremum.

c) Suppose you start at the point $\mathbf{p} = (1,1)$ in the (x,y)-plane and follow the contour of $f(x,y) = xy^2 + x^2y$ through $\mathbf{p}$ in one of the two possible directions. After some time you reach a point that is closer to $(0,0)$ than $\mathbf{p}$.

d) There exists a function $g\colon \mathbb{R}^2 \to \mathbb{R}$ with at least 2023 saddle points.

e) The set of all real numbers whose decimal expansion doesn't contain the digit 0 (i.e., only digits $1,2,3,4,5,6,7,8,9$ are allowed) has Lebesgue measure zero.

f) For any closed path γ in $\mathbb{R}^2$ and any choice of $a,b,c,d \in \mathbb{R}$ we have $\int_\gamma (ax+by)\,\mathrm{d}x + (cx+dy)\,\mathrm{d}y = \frac{c-b}{2}\int_\gamma x\,\mathrm{d}y - y\,\mathrm{d}x$.

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 + y^4 - 6\,x^2 - 4\,xy - 6\,y^2.$$

a) Which symmetry properties does f have? What can you conclude from this about the graph of f and the location/type of the critical points of f?

b) Determine all critical points of f and their types.

Hint: There are 9 critical points.

c) Does f have a global extremum?

Question 3 (ca. 7 marks)

Using the method of Lagrange multipliers, solve the optimization problem

$$\begin{aligned} \text{Maximize} \quad & \zeta = xy + 6yz + 6zx \\ \text{subject to} \quad & x^2 + y^2 + z^2 = 17. \end{aligned}$$

Note: Required are (i) a proof that the optimization problem has a solution, (ii) the optimal objective value ζ^*, and (iii) all optimal solutions (x^*, y^*, z^*).

Question 4 (ca. 6 marks)

Consider the function $F\colon (0,\infty) \to \mathbb{R}$ defined by

$$F(a) = \int_0^\infty \frac{\mathrm{d}x}{x^2 + a^2}.$$

a) Show that F is differentiable, and that $F'(a)$ can be obtained by differentation under the integral sign.

b) Using a), evaluate $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, $a > 0$.
Hint: The integral defining $F(a)$ can be evaluated using the substitution $x = at$.

Question 5 (ca. 7 marks)

a) Find the mass of the solid K in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad z^2 \leq 4x, \quad x^2 + y^2 \leq 16,$$

whose density is given by $\rho(x, y, z) = xyz^3$.

b) Find the area of the surface P in $\mathbb{R}^3$ consisting of all points (x, y, z) satisfying

$$z = x^{3/2} + y^{3/2}, \quad x \geq 0, \quad y \geq 0, \quad x + y \leq 2.$$

Solutions

1 a) True. The surface, call it S_A, is the 1-level set of $g_A(x,y,z) = x^3 + y^3 + z^3 + A\,xyz$, which has gradient $\nabla g_A(x,y,z) = (3x^2 + A\,yz, 3y^2 + A\,xz, 3z^2 + A\,xy)$.

For the proof suppose $(x,y,z) \in S_A$ satisfies $\nabla g(x,y,z) = (0,0,0)$. If $x = 0$ then the 2nd and 3rd coordinate of $\nabla g(x,y,z)$ are $3y^2$ resp. $3z^2$, so that $y = z = 0$ as well. But $(0,0,0) \notin S_A$, contradiction. Thus $x \neq 0$ and, by symmetry, $y \neq 0$ and $z \neq 0$. Further, $27x^2y^2z^2 = (3x^2)(3y^2)(3z^2) = (-Ayz)(-Axz)(-Axy) = -A^3x^2y^2z^2$, which together with $xyz \neq 0$ gives $A = -3$ as the only possible exception. In this case $\nabla g(x,y,z) = (0,0,0)$ reduces to $x^2 = yz$, $y^2 = xz$, $z^2 = xy$. From the 1st equation, $z = x^2/y$. Substituting this into the 2nd equation, $y^2 = x^3/y$ and hence $x^3 = y^3$, $x = y$. Then, by symmetry, $x = y = z$. But $g_{-3}(x,x,x) = x^3 + x^3 + x^3 - 3x^3 = 0$, and hence $(x,x,x) \notin S_{-3}$. This is the final contradiction. $\boxed{2}$

b) True. If f is the all-zero function, the statement is trivially true. Otherwise we may suppose w.l.o.g. that $f(x_0,y_0) > 0$ for some $(x_0,y_0) \in \mathbb{R}^2$. By assumption, there exists $R > 0$ such that $f(x,y) < f(x_0,y_0)$ for all points (x,y) with $|(x,y)| > R$. Since f is continuous, f attains a maximum on the closed disk $\mathrm{B}_R(0,0)$, say in (x_1,y_1). Since $(x_0,y_0) \in \mathrm{B}_R(0,0)$, we obtain $f(x,y) < f(x_0,y_0) \leq f(x_1,y_1)$ for all points (x,y) outside $\mathrm{B}_R(0,0)$. Thus the maximum in (x_1,y_1) is global.

Remark: It is not true that such a function f must have global extrema of both kinds, e.g., $f(x,y) = 1/(1+x^2+y^2)$ has a global maximum but no global minimum.

c) False. We have

$$\nabla f(x,y) = \begin{pmatrix} y^2 + 2xy, \\ x^2 + 2xy \end{pmatrix},$$

$$\begin{vmatrix} y^2 + 2xy & x \\ x^2 + 2xy & y \end{vmatrix} = y^3 + 2xy^2 - x^3 - 2x^2y = (y-x)(x^2 + 3xy + y^2).$$

Thus moving along the 2-contour from $(1,1)$ means moving in direction NW or SE (since $\nabla f(x,y)$ points to NE in the 1st quadrant and the coordinate axes, which are part of the 0-contour, cannot be reached). At a point on the 2-contour closest to $(0,0)$ (such a point exists by the usual continuity-compactness argument) the gradient $\nabla f(x,y)$ must be orthogonal to (x,y), which is the case only for points on the line $y = x$. But except for the starting point $(1,1)$, no such point can be reached. $\boxed{2}$

d) True. An example is

$$f(x,y) = (x+y-1)(y-x)(y-2x)\cdots(y-2023\,x).$$

For $m \in \{1,2,\dots,2023\}$ the intersection point of the lines $x + y = 1$ and $y = mx$, viz. $\left(\frac{1}{m+1}, \frac{m}{m+1}\right)$, is a saddle point of f. In order to see this, with m fixed it suffices to consider $g(x,y) = (x+y-1)(y-mx) = y^2 - mx^2 + (1-m)xy - y + mx$ instead.

$$\begin{aligned} g_x &= -2mx + (1-m)y + m, \\ g_y &= 2y + (1-m)x - 1, \\ g_{xx} &= -2m, \\ g_{xy} &= 1 - m = g_{yx}, \\ g_{yy} &= 2. \end{aligned}$$

One finds that $\nabla g\left(\frac{1}{m+1},\frac{m}{m+1}\right)=(0,0)$ (this also follows from the fact that the 0-contour of g or f is not smooth there), and $\det\mathbf{H}_g(x,y)=-4m-(1-m)^2<0$. $\boxed{2}$

e) True. Denote this set by S, and let $S_0=S\cap[0,1)$. Among the $10^k-10^{k-1}=9\cdot 10^{k-1}$ positive integers with exactly k decimal digits, 9^k don't involve the digit 0. Scaling by 10^{-k}, the set of real numbers in $[0,1)$ not involving the digit 0 in the first k digits after the decimal point has Lebesgue measure at most $9^k/(9\cdot 10^{k-1})=\left(\frac{9}{10}\right)^{k-1}$. Since $\left(\frac{9}{10}\right)^{k-1}\to 0$ for $k\to\infty$, we can conclude that S_0 has Lebesgue measure zero; cf. the corresponding argument for Cantor's Ternary Set. But then S, which is contained in a countable union of translates of S_0, must have Lebesgue measure zero as well. $\boxed{2}$

f) True. Using linearity of the line integral $\int_\gamma\omega$ as a function of ω, the equation can be rewritten as

$$\int_\gamma\left(ax+by+\frac{c-b}{2}\,y\right)dx+\left(cx+dy-\frac{c-b}{2}\,x\right)\mathrm{d}y=0$$
$$\iff\quad\int\left(ax+\frac{b+c}{2}\,y\right)\mathrm{d}x+\left(\frac{b+c}{2}\,x+dy\right)\mathrm{d}y=0.$$

Denoting the latter integrand by $\omega=M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y$, we have $M_y=\frac{b+c}{2}=N_x$, i.e., ω is exact in $\mathbb{R}^2$ and hence $\int_\gamma\omega=0$. $\boxed{2}$

Remarks: No marks were assigned for answers without justification.

$$\sum_1=12$$

2 a) $f(-x,-y)=f(x,y)=f(y,x)$ for $(x,y)\in\mathbb{R}^2$ $\boxed{1}$
$\Longrightarrow$ G_f is symmetric with respect to the z-axis and the plane $x=y$. $\boxed{1}$
Alternatively, $f(y,x))=f(x,y)=f(-y,-x)$ for $(x,y)\in\mathbb{R}^2$, which says that G_f is symmetric with respect to the two planes $x=\pm y$ (and implies the symmetry with respect to the z-axis).
If (x_0,y_0) is a critical point of f, so are $(-x_0,-y_0)$, (y_0,x_0), and $(-y_0,-x_0)$, and all have the same type. $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned}
f&=x^4+y^4-6\,x^2-4\,xy-6\,y^2\\
f_x&=4\,x^3-12\,x-4\,y\\
f_y&=4\,y^3-12\,y-4\,x,\\
f_x+f_y&=4(x^3+y^3)-16\,x-16\,y=4(x+y)\left(x^2-xy+y^2-4\right),\\
f_x-f_y&=4(x^3-y^3)-8x+8y=4(x-y)\left(x^2+xy+y^2-2\right).
\end{aligned}$$

Then $\nabla f(x,y)=(0,0)$ if 2 of the 4 functions f_x,f_y,f_x+f_y,f_x-f_y vanish at (x,y).

Clearly $\mathbf{p}_0=(0,0)$ is critical. $\boxed{\frac{1}{2}}$
Assuming $(x,y)\neq(0,0)$, we distinguish three mutually exclusive cases:

Case 1: $x = y$ Here $f_x + f_y = 0$ gives $x^2 - x^2 + x^2 = 4$, and hence $x = \pm 2$. This yields the two critical points $\mathbf{p}_1 = (2,2)$, $\mathbf{p}_2 = (-2,-2)$. $\boxed{1}$

Case 2: $x = -y$ Here $f_x - f_y = 0$ gives $x^2 - x^2 + x^2 = 2$, and hence $x = \pm\sqrt{2}2$. This yields the two critical points $\mathbf{p}_3 = \big(\sqrt{2}, -\sqrt{2}\big)$, $\mathbf{p}_4 = \big(-\sqrt{2}, \sqrt{2}\big)$. $\boxed{1}$

Case 3: $x \neq \pm y$ Here we must have $x^2 - xy + y^2 = 4 \wedge x^2 + xy + y^2 = 2$. Adding/subtracting the two equations gives $2\,x^2 + 2\,y^2 = 6$, $-2xy = 2$, i.e., $x^2 + y^2 = 3 \wedge xy = -1$. $\Longrightarrow x^2 + (-1/x)^2 = 3$, i.e., $x^4 - 3\,x^2 + 1 = 0$, $x^2 = \frac{1}{2}\big(3 \pm \sqrt{5}\big)$, $x = \pm\frac{1}{2}\big(1 \pm \sqrt{5}\big)$. This yields the four critical points

$$\mathbf{p}_5 = \left(\tfrac{1+\sqrt{5}}{2}, \tfrac{1-\sqrt{5}}{2}\right),\quad \mathbf{p}_6 = \left(\tfrac{1-\sqrt{5}}{2}, \tfrac{1+\sqrt{5}}{2}\right),\quad \mathbf{p}_7 = \left(\tfrac{-1+\sqrt{5}}{2}, \tfrac{-1-\sqrt{5}}{2}\right),\quad \mathbf{p}_8 = \left(\tfrac{-1-\sqrt{5}}{2}, \tfrac{-1+\sqrt{5}}{2}\right).$$

$\boxed{2}$

When determining the types of the critical points, by a) we need only test $\mathbf{p}_0$ and one from each set $\{\mathbf{p}_1, \mathbf{p}_2\}$, $\{\mathbf{p}_3, \mathbf{p}_4\}$, $\{\mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7, \mathbf{p}_8\}$. We have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12\,x^2 - 12 & -4 \\ -4 & 12\,y^2 - 12 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_0) = \begin{pmatrix} -12 & -4 \\ -4 & -12 \end{pmatrix},\quad \mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 36 & -4 \\ -4 & 36 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_3) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix},\quad \mathbf{H}_f(\mathbf{p}_5) = \begin{pmatrix} 6 + 6\sqrt{5} & -4 \\ -4 & 6 - 6\sqrt{5} \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_0)$ is negative definite (determinant > 0, top-left entry < 0), the point $\mathbf{p}_0$ is a strict local maximum. $\boxed{\tfrac{1}{2}}$

Since $\mathbf{H}_f(\mathbf{p}_1)$, $\mathbf{H}_f(\mathbf{p}_3)$ are positive definite (determinant > 0, top-left entry > 0), the points $\mathbf{p}_1$, $\mathbf{p}_2, \mathbf{p}_3$, $\mathbf{p}_4$ are strict local minima. $\boxed{2}$

Since $\mathbf{H}_f(\mathbf{p}_5)$ is indefinite (determinant < 0), the points $\mathbf{p}_5$, $\mathbf{p}_6, \mathbf{p}_7$, $\mathbf{p}_8$ are saddle points. $\boxed{2}$

c) Yes. The points $\mathbf{p}_1$, $\mathbf{p}_2$ are global minima (with value $f(\mathbf{p}_i) = -32$).

The existence of a global minimum follows from $\lim_{|(x,y)|\to\infty} f(x,y) = +\infty$ using an argument analogous to that in the solution to Question 1 b). Indeed, from $x^2 + y^2 \geq 2xy$ we have $6x^2 + 4xy + 6y^2 \leq 8(x^2+y^2)$ and $x^4 + y^4 = (x^2+y^2)^2 - 2x^2y^2 \geq \frac{1}{2}(x^2+y^2)^2$, and hence

$$f(x,y) \geq \frac{1}{2}\,r^4 - 8\,r^2, \quad r = |(x,y)|\,.$$

This clearly implies $\lim_{|(x,y)|\to\infty} f(x,y) = +\infty$. $\boxed{1\tfrac{1}{2}}$

Since the global minima must be among the local minima, in order to find them we only need to compare the values of f at $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. The proof is then finished by computing $f(\mathbf{p}_1) = f(\mathbf{p}_2) = -32$, $f(\mathbf{p}_3) = f(\mathbf{p}_4) = -8$.

Since $f(x,0) = x^4 - 6\,x^2$ is unbounded from above, there is no global maximum. $\boxed{\tfrac{1}{2}}$

$$\sum_{2} = 14$$

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3 The continuous function $f(x,y,z)=xy+6yz+6zx$ attains a maximum on the sphere $\mathrm{B}_{\sqrt{17}}(0,0,0)$, which is closed and bounded. This shows that the optimization problem has at least one solution. $\boxed{1}$

Setting $g(x,y,z)=x^2+y^2+z^2$, the task is to minimize f on $\mathbb{R}^3$ under the constraint $g(x,y,z)=17$.

$$\nabla f(x,y,z)=(y+6z,x+6z,6x+6y),\qquad \nabla g(x,y,z)=(2x,2y,2z).$$

Since $\nabla g(x,y,z)\neq(0,0,0)$ for all points on the sphere $\mathrm{B}_{\sqrt{17}}(0,0,0)$, the theorem on Langrange multipliers yields that every optimal solution must satisfy $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ for some $\lambda\in\mathbb{R}$. This gives the system of equations

$$\begin{aligned} y+6z&=\lambda x,\\ x+6z&=\lambda y,\\ 6x+6y&=\lambda z,\\ x^2+y^2+z^2&=17. \end{aligned} \qquad \boxed{3}$$

(For simplicity we have replaced λ by $\lambda/2$.)

The solutions (x,y,z,λ) of this system are precisely the unit eigenvectors of the matrix

$$\mathbf{A}=\begin{pmatrix}0&1&6\\1&0&6\\6&6&0\end{pmatrix}$$

together with the corresponding eigenvalues.

Since $\chi_{\mathbf{A}}(X)=X^3-73\,X-72=(X+1)(X^2-X-72)=(X+1)(X+8)(X-9)$, the eigenvalues of $\mathbf{A}$ are $\lambda_1=9$, $\lambda_2=-1$, $\lambda_3=-8$. This shows already that the eigenspaces of $\mathbf{A}$ are one-dimensional and that the above system has exactly 6 solutions. Next we compute the corresponding eigenvectors. Unit eigenvectors will be denoted by $\mathbf{u}_i$ and eigenvectors of length $\sqrt{17}$ by $\mathbf{v}_i$.

$\underline{\lambda_1=9:}$

$$\mathbf{A}-9\,\mathbf{I}=\begin{pmatrix}-9&1&6\\1&-9&6\\6&6&-9\end{pmatrix}\to\begin{pmatrix}1&-9&6\\0&-80&60\\0&60&-45\end{pmatrix}\to\begin{pmatrix}1&-9&6\\0&-4&3\\0&0&0\end{pmatrix}$$

$\Longrightarrow \mathbf{u}_1=\pm\frac{1}{\sqrt{34}}\,(3,3,4)^{\mathsf{T}}$, $\mathbf{v}_1=\pm\frac{1}{\sqrt{2}}\,(3,3,4)^{\mathsf{T}}$; $\boxed{1}$

$\underline{\lambda_2=-1:}$

$$\mathbf{A}+\mathbf{I}=\begin{pmatrix}1&1&6\\1&1&6\\6&6&1\end{pmatrix}\to\begin{pmatrix}1&1&6\\0&0&-35\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&1&6\\0&0&1\\0&0&0\end{pmatrix}$$

$\Longrightarrow \mathbf{u}_2=\pm\frac{1}{\sqrt{2}}\,(1,-1,0)^{\mathsf{T}}$, $\mathbf{v}_2=\pm\frac{\sqrt{17}}{\sqrt{2}}\,(1,-1,0)^{\mathsf{T}}$; $\boxed{1}$

$\underline{\lambda_3=-8:}$

$$\mathbf{A}+8\,\mathbf{I}=\begin{pmatrix}8&1&6\\1&8&6\\6&6&8\end{pmatrix}\to\begin{pmatrix}1&8&6\\0&-63&-42\\0&-42&-28\end{pmatrix}\to\begin{pmatrix}1&8&6\\0&3&2\\0&0&0\end{pmatrix}$$

$\Longrightarrow \mathbf{u}_3 = \pm\frac{1}{\sqrt{17}}\,(2,2,-3)^\mathsf{T}$, $\mathbf{v}_3 = \pm(2,2,-3)^\mathsf{T}$. 1

Since $f(3,3,4) = 153 > 0$, $f(1,-1,0) = -1 < 0$, $f(2,2,-3) = -68 < 0$, the maximum is attained at $(x^*,y^*,z^*) = \pm\frac{1}{\sqrt{2}}\,(3,3,4)^\mathsf{T}$, and $\zeta^* = 153/2$. 2

$$\sum_3 = 9$$

4 a) Writing $f(x,a) = \frac{1}{x^2+a^2}$, we have

$$\int_0^\infty f_a(x,a)\,\mathrm{d}x = \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,\mathrm{d}x\,. \qquad \boxed{1}$$

Since $x^2+a^2 \geq 2xa$, we can bound the integrand as follows:

$$|f_a(x,a)| = \frac{2a}{(x^2+a^2)^2} \leq \frac{2a}{2xa(x^2+a^2)} = \frac{1}{x(x^2+a^2)} \leq \frac{1}{x(x^2+\delta^2)} =: \Phi(x), \qquad \boxed{2}$$

provided that $a \geq \delta > 0$. Since $\Phi(x)$ is independent of a and integrable over $[0,\infty)$, this shows that F is differentiable in (δ,∞) and $F'(a)$ can be obtained by differentation under the integral sign. Letting $\delta \downarrow 0$, we then obtain the assertion for the whole domain $(0,\infty)$. 1

b) From a) we have

$$F'(a) = \int_0^\infty \frac{-2a}{(x^2+a^2)^2}\,\mathrm{d}x, \quad \text{i.e.,} \quad \int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = -\frac{F'(a)}{2a}. \qquad \boxed{1}$$

On the other hand we have

$$\begin{aligned} F(a) &= \int_0^\infty \frac{\mathrm{d}x}{x^2+a^2} \\ &= a\int_0^\infty \frac{\mathrm{d}t}{a^2t^2+a^2} && \text{(Subst. } x = at,\ \mathrm{d}x = a\,\mathrm{d}t) \\ &= \frac{1}{a}\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \frac{1}{a}\left[\arctan t\right]_0^\infty = \frac{\pi}{2a}. && \boxed{1} \end{aligned}$$

It follows that

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = -\frac{-\pi/2a^2}{2a} = \frac{\pi}{4a^3}. \qquad \boxed{1}$$

$$\sum_4 = 7$$

5 a) The mass of K is

$$\begin{aligned}
m &= \int_K xyz^3 \, \mathrm{d}^3(x,y,z) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} \int_{z=0}^{2\sqrt{x}} xyz^3 \, \mathrm{d}z \, \mathrm{d}^2(x,y) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} xy \left[\frac{z^4}{4}\right]_{z=0}^{2\sqrt{x}} \mathrm{d}^2(x,y) \\
&= \int\limits_{\substack{x^2+y^2\leq 16\\ x,y\geq 0}} 4\,x^3 y \, \mathrm{d}^2(x,y) \\
&= 4 \int\limits_{\substack{0\leq r\leq 4\\ 0\leq\theta\leq\pi/2}} (r\cos\theta)^3 r\sin\theta \, r \, \mathrm{d}^2(r,\theta) \\
&= 4 \int_0^4 r^5 \, \mathrm{d}r \int_0^{\pi/2} \cos^3\theta \sin\theta \, \mathrm{d}\theta \\
&= \frac{4^7}{6}\left[-\frac{1}{4}\cos^4\theta\right]_0^{\pi/2} = \frac{4^6}{6} = \frac{2^{11}}{3} = \frac{2048}{3}.
\end{aligned}$$

3

b) Denoting the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, $(0,2)$ by Δ, the surface P is the graph of $f(x,y) = x^{3/2} + y^{3/2}$, $(x,y) \in \Delta$. Using the formula for such surfaces, or going the long way using the parametrization $\gamma(x,y) = \big(x,y,f(x,y)\big)$, we obtain the surface area as

$$\begin{aligned}
A &= \int_\Delta \sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y) = \int_\Delta \sqrt{1+\left|\tfrac{3}{2}\left(\sqrt{x},\sqrt{y}\right)\right|^2}\,\mathrm{d}^2(x,y)\\
&= \int_\Delta \sqrt{1+\tfrac{9}{4}(x+y)}\,\mathrm{d}^2(x,y) && \boxed{1}\\
&= \frac{1}{2}\int_0^2\int_0^{2-x}\sqrt{4+9x+9y}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{1}{2}\int_0^2\left[\frac{2}{27}(4+9x+9y)^{3/2}\right]_{y=0}^{2-x}\mathrm{d}x\\
&= \frac{1}{27}\int_0^2 22^{3/2}-(4+9x)^{3/2}\,\mathrm{d}x\\
&= \frac{1}{27}\left(2\cdot 22^{3/2}-\left[\frac{2}{45}(4+9x)^{5/2}\right]_0^2\right)\\
&= \frac{2}{27}\,22^{3/2}-\frac{2}{27\cdot 45}\left(22^{5/2}-4^{5/2}\right)\\
&= \frac{64+46\cdot 22\sqrt{22}}{27\cdot 45}\\
&= \frac{64+1012\sqrt{22}}{1215} && \boxed{3}
\end{aligned}$$

$$\sum_{5} = 7$$

$$\sum_{\text{Final Exam}} = 12+14+9+7+7 = 49 = 40+9$$