Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The surface in $\mathbb{R}^3$ with equation $x^2y + y^2z + z^2x = 1$ is smooth.

b) Suppose you start at the point $(1, 1)$ in the (x, y)-plane and follow (and continuously adjust) the direction of steepest ascent of $f(x, y) = \frac{x-y}{x+y}$. After some time (provided you won't get tired) you will cross the x-axis at a point $(x_0, 0)$ with $x_0 > 2$.

c) If $f\colon [0, 1] \to \mathbb{R}$ is continuous and $f(0) = 0$ then $\lim_{n\to\infty} \int_0^1 f(x^n)\,\mathrm{d}x = 0$.

d) The equation $x^2 + xy + y^2 = 3$ defines a circle, which is symmetric to the line $y = x$.

e) There exists a subset D of the upper half plane $\{(x, y) \in \mathbb{R}^2; y > 0\}$ whose set of accumulation points is equal to the real axis (x-axis).

f) If γ is a closed path in $\mathbb{R}^3$ satisfying $\int_\gamma x\,\mathrm{d}y + y\,\mathrm{d}z + z\,\mathrm{d}x = 0$, we must have $\int_\gamma x\,\mathrm{d}z + y\,\mathrm{d}x + z\,\mathrm{d}y = 0$ as well.

Question 2 (ca. 12 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^4 - x^3y - xy + y^2.$$

a) Which obvious symmetry property does f have? What can you conclude from this about the graph and the contours of f?

b) Determine all critical points of f and their types.

Hint: There are 5 critical points.

c) Does f have a global extremum?

d) Determine the extrema of f on the unit square $Q = \{(x, y) \in \mathbb{R}^2; 0 \leq x, y \leq 1\}$.

Question 3 (ca. 12 marks)

The sphere $x^2 + y^2 + z^2 = 9$ intersects the surface $xy + yz + zx = 8$ in the first octant $O = \{(x, y, z) \in \mathbb{R}^3; x, y, z \geq 0\}$ in a curve C.

a) Show that C has no points on the boundary of O.

b) Show that there exist points on C with minimum, resp., maximum height (z-coordinate).

Hint: $(2, 2, 1) \in C$.

c) Using the method of Lagrange multipliers on the interior of O, determine all those points.

Note: Don't forget to check for points on C where the Jacobi matrix of the vectorial constraint doesn't have full row rank. In fact there are no such points, but this requires a proof.

d) At the point $(2,2,1)$ the curve C admits locally a parametrization $\gamma(x) = \bigl(x, h(x), k(x)\bigr)$ with functions $h, k \colon (2-\epsilon, 2+\epsilon) \to \mathbb{R}$. Determine $h'(2)$ and $k'(2)$.

Question 4 (ca. 12 marks)

Consider the transformation

$$T(s,t,u) = (us\cos t, us\sin t, us + ut)$$

from the region $U = \bigl\{(s,t,u) \in \mathbb{R}^3; 0 < s < t < 2\pi, 0 < u < 1\bigr\}$ to the region $V = T(U) \subset \mathbb{R}^3$, and the "helicoid"

$$S = \bigl\{(s\cos t, s\sin t, s+t); 0 < s < t < 2\pi\bigr\}$$

bounding V from above.

a) Show that $T \colon U \to V$ is a diffeomorphism (i.e., T is one-to-one, and both T and T^{-1} are differentiable).

b) Determine the volume of V.

c) Express the surface area of S as an ordinary 1-dimensional Riemann integral.

Solutions

1 a) True. The surface is a level set of $g(x,y,z)=x^2y+y^2z+z^2x$, which has gradient $\nabla g(x,y,z)=(2xy+z^2,2yz+x^2,2zx+y^2)$. If $\nabla g(x,y,z)=(0,0,0)$ then $8x^2y^2z^2=(2xy)(2yz)(2zx)=(-x^2)(-y^2)(-z^2)=-x^2y^2z^2$, and hence $xyz=0$. By symmetry, we can assume $x=0$. Then $g_x=0$ gives $z=0$, and $g_z=0$ gives $y=0$. But the point $(0,0,0)$ isn't on the surface, and hence $\nabla g(x,y,z)=(0,0,0)$ has no solution on the surface. $\boxed{2}$

b) False. We have

$$f_x=\frac{1(x+y)-1(x-y)}{(x+y)^2}=\frac{2y}{(x+y)^2},$$
$$f_y=\frac{(-1)(x+y)-1(x-y)}{(x+y)^2}=\frac{-2x}{(x+y)^2},$$

and hence gradients $\nabla f(x,y)$ in the open first quadrant point south-eastern and are orthogonal to (x,y). Hence, following the direction of steepest ascent (i.e., the gradient) you will move south-eastern from $(1,1)$, turn more and more southward, and cross the x-axis at a point closer to the origin than when following the gradient at $(1,1)$ all the time, i.e., closer than the point $(2,0)$. $\boxed{2}$

c) True. The function sequence $f_n(x)=f(x^n)$, defined on $[0,1]$, converges point-wise to the all-zero function, except possibly for $x=1$, where the limit is $f(1)$. For this observe that $x^n\to 0$ and hence, since f is continuous, $f(x^n)\to f(0)=0$ for $0\leq x<1$. The functions f_n are continuous, hence Lebesgue-integrable, and bounded by a constant $M>0$ independently of n. (Any bound for the continuous function f works also for f_n. Thus $\Phi(x)=M$, $x\in[0,1]$, serves as an integrable bound for (f_n) and allows us to apply Lebesgue's dominated convergence theorem to conclude that $\lim_{n\to\infty}\int_0^1 f(x^n)\,\mathrm{d}x=\lim_{n\to\infty}\int f_n(x)\,\mathrm{d}x=\int_0^1\lim_{n\to\infty}f_n(x)\,\mathrm{d}x=\int_0^1 0\,\mathrm{d}x=0$. $\boxed{2}$

d) False. It is an ellipse with semi-axes $a=\sqrt{6}$ on the line $y=-x$, and $b=\sqrt{2}$ on the line $y=x$. This can be seen by diagonalizing the corresponding symmetric matrix, which is $\begin{pmatrix}1&1/2\\1/2&1\end{pmatrix}$ and has eigenvalues $3/2$ and $1/2$. But to prove the statement false, it suffices to observe that $(1,1)$ and $\left(-\sqrt{3},\sqrt{3}\right)$ satisfy the equation and have different distance from the center $(0,0)$. $\boxed{2}$

e) True. Enumerate the rational numbers as $q_1,q_2,q_3,\dots$ and define $\mathbf{x}^{(n)}=(q_n,1/n)$ for $n\in\mathbb{N}$. We claim that the sequence $\left(\mathbf{x}^{(n)}\right)$, or its range D, has $D'=\mathbb{R}\times\{0\}$. Clearly no point (x,y) with $y<0$ can be in D'. If $y>0$, the disk around (x,y) with radius $y/2$ contains only finitely many points in D (it can contain only points $\mathbf{x}^{(n)}$ with $n<2/y$), showing that (x,y) cannot be in D' either. Now consider a point $(x,0)$ and a disk B of radius $\epsilon>0$ around this point. Since $\mathbb{Q}'=\mathbb{R}$, there exist infinitely many n such that $|x-q_n|<\epsilon/\sqrt{2}$. Of these all but finitely many also have $1/n<\epsilon/\sqrt{2}$. Since B contains all points $\mathbf{x}^{(n)}$ with n satisfying both conditions, it contains infinitely many points of D, i.e., we have shown $(x,0)\in D'$. $\boxed{2}$

f) True. The sum of the two integrands is $(y+z)\,\mathrm{d}x+(x+z)\,\mathrm{d}y+(x+y)\,\mathrm{d}z$, which is exact in $\mathbb{R}^3$. Hence the sum of the two integrals is zero (since it is the integral of the sum). So, if one integral is zero, the other must be too. $\boxed{2}$

Remarks: No marks were assigned for answers without justification.

$$\sum_1 = 12$$

2 a) $f(-x,-y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$
$\Longrightarrow$ The graph of f is symmetric with respect to the z-axis. $\boxed{1}$
The contours of f are point-symmetric with respect to the origin. $\boxed{1}$

b) Using the shorthands f, f_x, f_y for $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, we compute

$$\begin{aligned} f_x &= 4\,x^3 - 3\,x^2y - y, \\ f_y &= -x^3 - x + 2\,y, \end{aligned} \qquad \boxed{1}$$

$$\begin{aligned} \nabla f(x,y) = (0,0) &\Longrightarrow y = \tfrac{1}{2}\left(x^3+x\right) \Longrightarrow 4x^3 - \left(3x^2+1\right)\tfrac{1}{2}\left(x^3+x\right) = 0 \\ &\Longrightarrow 3x^5 - 4x^3 + x = 0 \\ &\Longrightarrow x(x^2-1)(3x^2-1) = 0. \end{aligned}$$

$\Longrightarrow$ The critical points of f are

$$\begin{gathered} \mathbf{p}_1 = (0,0), \quad \mathbf{p}_2 = (1,1), \quad \mathbf{p}_3 = (-1,-1), \\ \mathbf{p}_4 = \left(\tfrac{1}{3}\sqrt{3}, \tfrac{2}{9}\sqrt{3}\right), \quad \mathbf{p}_5 = \left(-\tfrac{1}{3}\sqrt{3}, -\tfrac{2}{9}\sqrt{3}\right). \end{gathered} \qquad \boxed{2\tfrac{1}{2}}$$

Further we have

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 - 6xy & -3x^2-1 \\ -3x^2-1 & 2 \end{pmatrix},$$

$$\mathbf{H}_f(\mathbf{p}_1) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{2/3}) = \begin{pmatrix} 6 & -4 \\ -4 & 2 \end{pmatrix}, \quad \mathbf{H}_f(\mathbf{p}_{4/5}) = \begin{pmatrix} 8/3 & -2 \\ -2 & 2 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$ has determinant $-1 < 0$, the point $\mathbf{p}_1$ is a saddle point. $\boxed{\tfrac{1}{2}}$
Since $\mathbf{H}_f(\mathbf{p}_{2/3})$ has determinant $12-16 = -4 < 0$, the points $\mathbf{p}_2$, $\mathbf{p}_3$ are saddle points. $\boxed{1}$
Since $\mathbf{H}_f(\mathbf{p}_{4/5})$ is positive definite ($f_{xx}(\mathbf{p}_{4/5}) = 8/3 > 0$, $\det \mathbf{H}_f(\mathbf{p}_{4/5}) = 16/3 - 4 = 4/3 > 0$), the points $\mathbf{p}_4$, $\mathbf{p}_5$ are strict local minima. $\boxed{1}$
The corresponding value is $f(\mathbf{p}_{4/5}) = -1/27$.

c) No. This follows, e.g., from $f(x,0) = x^4 \to +\infty$ for $x \to \pm\infty$, $f(x,2x) = -x^4 + 2x^2 \to -\infty$ for $x \to \pm\infty$. $\boxed{1}$

d) Extrema located in Q° must be critical points, and hence equal to $\mathbf{p}_4$. On the boundary ∂Q we have

$$\begin{aligned} &f(x,0) = x^4, \\ &f(0,y) = y^2, \\ &f(x,1) = x^4 - x^3 - x + 1 = (x^3-1)(x-1), \\ &f(1,y) = 1 - 2y + y^2 = (1-y)^2. \end{aligned}$$

One sees that the values on ∂Q vary between 0 and 1, with 1 attained at $(1,0)$, $(0,1)$, and 0 attained at $(0,0)$, $(1,1)$. Comparing these with $f(\mathbf{p}_4) = -1/27$ shows that f on Q attains its minimum at $\mathbf{p}_4$ and two maxima at $(1,0)$, $(0,1)$. $\boxed{3}$

Remarks:

$$\sum_2 = 12$$

3 a) Suppose C contains a point with $x = 0$. Then $y^2 + z^2 = 9$, $yz = 8$, and hence $(y-z)^2 = y^2 + z^2 - 2yz = -7 < 0$, contradiction. Then, by symmetry, C doesn't contain a point with $y = 0$ or $z = 0$ either. $\boxed{1}$

b) Using the hint, C is non-empty; C is closed as the intersection of two level sets of continuous functions and the closed set O; and C is bounded as a subset of a sphere. Hence the continuous function $f(x,y,z) = z$ attains a minimum and a maximum on C. $\boxed{2}$

c) By a), C is contained in O°, so that we can apply the method of Lagrange multipliers to the objective function f and the two constraints $g_1(x,y,z) = x^2 + y^2 + z^2 - 9$, $g_2(x,y,z) = xy + yz + zx - 8$, all with domain O°, to find those points. For $\mathbf{g} = (g_1, g_2)^\mathsf{T}$ we have

$$\mathbf{J_g}(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \\ y+z & x+z & x+y \end{pmatrix}.$$

Points on C where $\mathbf{J_g}$ has rank < 2 would need to be checked separately, but there are no such points as we now show:

$\operatorname{rank}(\mathbf{J_g}) < 2$ implies that all three 2×2 subdeterminants of $\mathbf{J_g}$ vanish. In particular we then have $x(x+z) = y(y+z)$, i.e., $x^2 - y^2 + xz - yz = 0$, which can be factorized as $(x-y)(x+y+z) = 0$. Since the 2nd factor is positive on O°, we must have $x = y$. By symmetry (or using the other two subdeterminants), we also have $x = z$ and $y = z$, i.e., $x = y = z$. But, since $3x^2 = 9$ and $3x^2 = 8$ are mutually exclusive, C doesn't contain a point with $x = y = z$. $\boxed{2}$

Thus the Lagrange multiplier condition applies to any minimum/maximum of f on C and yields the equations:

$$\begin{aligned} \lambda x + \mu(y+z) &= 0, \\ \lambda y + \mu(x+z) &= 0, \\ \lambda z + \mu(x+y) &= 1, \\ x^2 + y^2 + z^2 &= 9, \\ xy + yz + zx &= 8. \end{aligned}$$ $\boxed{2}$

Since $x, y, z > 0$, the multipliers λ, μ must be nonzero. Then, from the first two equations we obtain as above $x = y$. $\boxed{1}$

This leaves the two equations $2x^2 + z^2 = 9$, $x^2 + 2xz = 8$. Solving the 2nd equation for z and substituting the result into the 1st equation gives

$$2\,x^2 + \left(\frac{8 - x^2}{2x}\right)^2 = 9 \iff 8\,x^4 + (8 - x^2)^2 = 36\,x^2$$
$$\iff 9\,x^4 - 52\,x^2 + 64 = 0 \iff 9(x^2 - 4)(x^2 - 16/9) = 0.$$

Thus $x = 2 \vee x = 4/3$, giving the two points $(2,2,1)$, $\left(\frac{4}{3},\frac{4}{3},\frac{7}{3}\right)$. $\boxed{2}$

Thus $(2,2,1)$ is the unique point of minimal height 1 on C, and $\left(\frac{4}{3},\frac{4}{3},\frac{7}{3}\right)$ is the unique point of maximal height $\frac{7}{3}$ on C. $\boxed{1}$

d) The vector $\gamma'(2) = \big(1, h'(2), k'(2)\big)$ gives the tangent direction to C in $(2,2,1)$. The tangent direction is orthogonal to $\nabla g_1(2,2,1) = (2,2,2)$ and $\nabla g_2(2,2,1) = (3,3,4)$ and hence equal to $\mathbb{R}(1,-1,0)$. Thus $h'(2) = -1$, $k'(2) = 0$. $\boxed{1}$

Remarks: It is possible to solve the question without Lagrange multipliers, noting that $(x+y+z)^2 = x^2+y^2+z^2+2(xy+xz+yz) = 9+2\cdot 8 = 25$ and hence C is the intersection of the sphere and the plane $x+y+z=5$. But then an independent argument must be given that the maximum and minimum must satisfy $x=y$. On can also use Lagrange multipliers with the constraints $x^2+y^2+z^2=9$ and $x+y+z=5$. If no Lagrange multipliers were used at all, 2 marks were subtracted (the marks for the set of 5 equations). The statement of the question clearly says that Lagrange multipliers must be used.

$$\sum_3 = 12$$

4 a) Suppose $T(s,t,u) = T(s',t',u')$, i.e.,

$$(us\cos t, us\sin t, us+ut) = (u's'\cos t', u's'\sin t', u's'+u't').$$

Looking at the first two coordinates, which for fixed s,u, resp., s', u' parametrize a circle of radius $us>0$, we obtain $t=t'$ since $t,t' \in (0,2\pi)$, and $us = u's'$ since $u,s,u',s'>0$). Then, looking at the last coordinate we find $ut = u't' = u't$ and hence $u = u'$ (since $t>0$). Thus $us = u's' = us'$, implying $s = s'$. $\boxed{2}$

Clearly T is continuously differentiable with

$$\mathbf{J}_T(s,t,u) = \begin{pmatrix} u\cos t & -us\sin t & s\cos t \\ u\sin t & us\cos t & s\sin t \\ u & u & s+t \end{pmatrix}. \qquad \boxed{1}$$

For the differentiability of T^{-1} it suffices to show that $\mathbf{J}_T(s,t,u)$ is invertible on U. $\boxed{1}$

We have

$$\begin{aligned}
\det \mathbf{J}_T(s,t,u) &= u^2 \begin{vmatrix} \cos t & -s\sin t & s\cos t \\ \sin t & s\cos t & s\sin t \\ 1 & 1 & s+t \end{vmatrix} \\
&= u^2 \begin{vmatrix} \cos t & -s\sin t & s\cos t \\ \sin t & s\cos t & s\sin t \\ 1 & 1 & s \end{vmatrix} + u^2 \begin{vmatrix} \cos t & -s\sin t & 0 \\ \sin t & s\cos t & 0 \\ 1 & 1 & t \end{vmatrix} \\
&= u^2 \begin{vmatrix} \cos t & -s\sin t & 0 \\ \sin t & s\cos t & 0 \\ 1 & 1 & t \end{vmatrix} = u^2 t \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} = u^2 st \neq 0
\end{aligned}$$

for $(s,t,u) \in U$. This implies that $\mathbf{J}_T(s,t,u)$ is invertible on U. $\boxed{2}$

b) Applying the change-of-variables formula to T (valid on account of a)) gives

$$\begin{aligned}
\mathrm{vol}_2(V) &= \int_{T(U)} 1\,\mathrm{d}^3\mathbf{y} = \int_U |\mathbf{J}_T(s,t,u)|\,\mathrm{d}^3(s,t,u) && \boxed{1}\\
&= \int_0^{2\pi}\int_0^t\int_0^1 u^2 st\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{3}\int_0^{2\pi}\int_0^t st\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{1}{6}\int_0^{2\pi} t^3\,\mathrm{d}t\\
&= \frac{1}{6}\left[t^4/4\right]_0^{2\pi} = \frac{16}{24}\,\pi^4 = \frac{2}{3}\,\pi^4. && \boxed{2}
\end{aligned}$$

c) The (non-standard) helicoid is parametrized by $\gamma(s,t) = (s\cos t, s\sin t, s+t)$, $0 < s < t < 2\pi$.

$$\begin{aligned}
\mathbf{J}_\gamma(t) &= \begin{pmatrix} \cos t & -s\sin t\\ \sin t & s\cos t\\ 1 & 1\end{pmatrix},\\
\mathbf{J}_\gamma(t)^\mathsf{T}\mathbf{J}_\gamma(t) &= \begin{pmatrix} 2 & 1\\ 1 & 1+s^2\end{pmatrix},\\
g_\gamma(s,t) &= 2(1+s^2) - 1 = 1 + 2s^2,\\
\sqrt{g_\gamma(s,t)} &= \sqrt{1+2s^2}, && \boxed{2}\\
\mathrm{vol}_2(S) &= \int\limits_{\substack{(s,t)\in\mathbb{R}^2\\ 0<s<t<2\pi}} \sqrt{1+2s^2}\,\mathrm{d}^2(s,t)\\
&= \int_0^{2\pi}\int_0^t \sqrt{1+2s^2}\,\mathrm{d}s\,\mathrm{d}t\\
&= \int_0^{2\pi}\int_s^{2\pi} \sqrt{1+2s^2}\,\mathrm{d}t\,\mathrm{d}s && \text{(Fubini)}\\
&= \int_0^{2\pi} (2\pi - s)\sqrt{1+2s^2}\,\mathrm{d}s\,. && \boxed{2}
\end{aligned}$$

Remarks:

$$\sum_4 = 13$$

$$\sum_{\text{Final Exam}} = 40 + 9$$