- Vector Field

1 Definitio Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

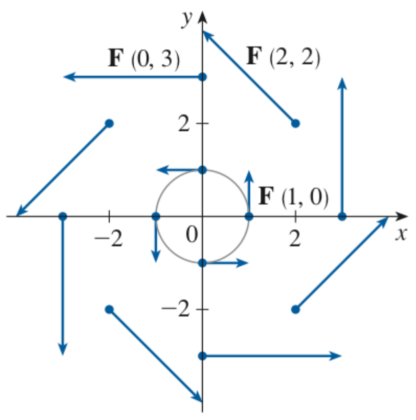


FIGURE 5
$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$

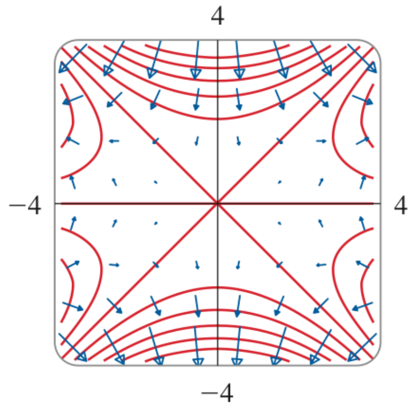


FIGURE 15

W38. Determine a potential function of w.

$f = 3x + x^2y - y^3 + C$

textbook P1149

EXAMPLE 3 If $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

f_x $\quad$ f_y

$f_x = (3 + 2xy)$

$f = 3x + x^2y + f(y)$

$f_y = x^2 + f'(y) = x^2 - 3y^2$

$f(y) = -y^3 + C$

$f = 3x + x^2y - y^3 + C$

- Line Integral

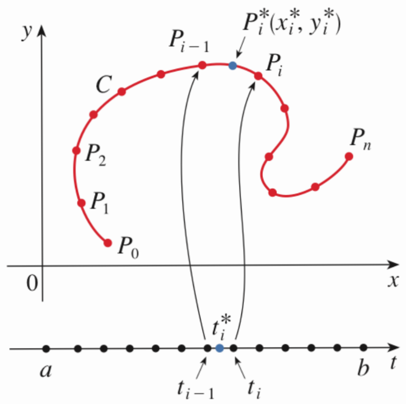


FIGURE 1

2 Definitio If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

↓

3 $$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{ds}$$

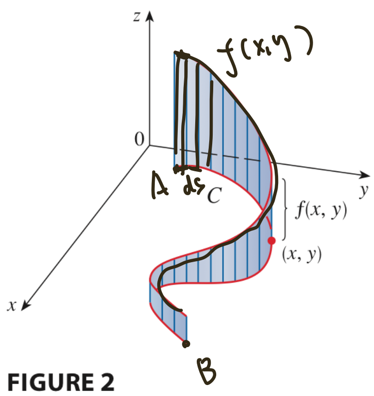


FIGURE 2

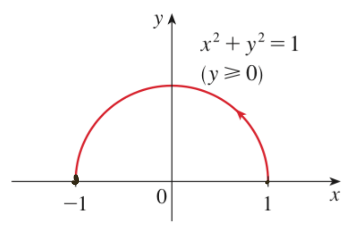


FIGURE 3

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

$x = \cos t$

$y = \sin t \qquad 0 \leq t \leq \pi$

$$\int_0^\pi 2 + \cos^2 t \sin t \cdot \sqrt{\cancel{\cos^2 t + \sin^2 t}} \cdot dt$$

$$2t - \frac{1}{3}\cos^3 t \Big]_0^\pi = 2\pi + \frac{2}{3}$$

- Exact: $\omega = (2x - z)\,dx + (y - xz)\,dy - xy\,dz$

Theorem

Suppose $\omega\colon D \to (\mathbb{R}^n)^*$, $D \subseteq \mathbb{R}^n$, is continuous and exact, $\omega = \mathrm{d}f$, and $\gamma\colon [a, b] \to D$ is any path. Then we have

$$\int_\gamma \omega = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Proof.

得到的2条推论

Corollary

Suppose $\omega\colon D \to (\mathbb{R}^n)^*$, $D \subseteq \mathbb{R}^n$, is continuous and exact.

1. *The integrals $\int_\gamma \omega$ are independent of path, i.e., for any two paths γ_1, γ_2 in D whose starting points and end points coincide we have $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.*
2. *For every closed path γ in D we have $\int_\gamma \omega = 0$.*

The Converse of the Corollary

Independence of path implies exactness

First observe/recall the following:

- If $\omega\colon D \to (\mathbb{R}^n)^*$, $D \subseteq \mathbb{R}^n$, is exact then D must be *open*, because $\omega(\mathbf{x}) = \mathrm{d}f(\mathbf{x})$ for $\mathbf{x} \in D$ implies in particular $D = D^\circ$.
- If in addition D is path-connected then any two antiderivatives f_1, f_2 of ω differ by a constant, since $\omega = \mathrm{d}f_1 = \mathrm{d}f_2$ implies $\mathrm{d}(f_1 - f_2) = 0$ on D, and we have seen that in this case $f_1 - f_2$ must be constant.

If ω is continuous, exact $\Longleftrightarrow$ independent of path

Locally Exact 1-Forms

Definition

A differential 1-form $\omega\colon D \to (\mathbb{R}^n)^*$, $D \subseteq \mathbb{R}^n$, is *locally exact* if every point $\mathbf{x} \in D$ has a neighborhood, on which ω is exact.

Of course we can assume that this neighborhood is a ball $\mathrm{B}_r(\mathbf{x})$ for some $r > 0$. If ω is exact on D then it is locally exact with D serving as the desired neighborhood for all $\mathbf{x} \in D$.

Proposition

If $\omega = \sum_{i=1}^n f_i \,\mathrm{d}x_i$ is locally exact (in particular, if ω is exact) and continuously differentiable (on D) then

$$(f_i)_{x_j} = (f_j)_{x_i} \quad \text{for all } 1 \le i < j \le n \quad \text{(on } D\text{)}.$$

For example, for $n=2$, $\omega = \underbrace{(3+2xy)}_{f_1}dx + \underbrace{(x^2-3y^2)}_{f_2}dy$

$n=3$. $\quad \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \quad 2x = 2x$.

$\_x \quad \_y \quad \_z$

exact $\Rightarrow (f_i)_{x_j} = (f_j)_{x_i}$

$+$ D is star-shaped

Theorem (Poincaré's Lemma)

Let $\omega = \sum_{i=1}^n f_i \,\mathrm{d}x_i$ be a continuously differentiable differential 1-form on $D \subseteq \mathbb{R}^n$. If ω satisfies $(f_i)_{x_j} = (f_j)_{x_i}$ for $1 \le i < j \le n$ and D is star-shaped then ω is exact.

Star-shaped:

Definition

A subset D of $\mathbb{R}^n$ is said to be *star-shaped* of there exists a "central" point $\mathbf{c} \in D$ such that for any other point $\mathbf{x} \in D$ the line segment $[\mathbf{c}, \mathbf{x}] = \{(1-t)\mathbf{c} + t\mathbf{x};\, 0 \le t \le 1\}$ is contained in D.

It will be shown that locally exact continuous differential 1-forms on a simply connected set $D \subseteq \mathbb{R}^n$ are exact. This vastly generalizes Poincaré's Lemma.

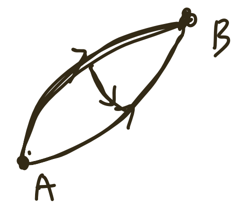
locally exact + simply-connected $\Rightarrow$ exact

W39.

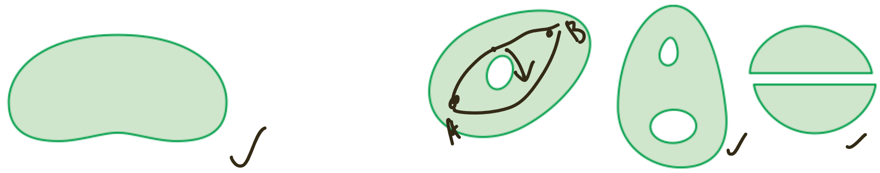
Simply Connected regions

Roughly speaking, a connected open set $D \subseteq \mathbb{R}^n$ is "simply connected" if every closed path in D can be continuously contracted to a point entirely within D. Subsets of $\mathbb{R}^2$ with this property must not contain holes.

：在定义域内，任意2条连接2个点的线可以无障碍地收缩到一起.



For R^2 :

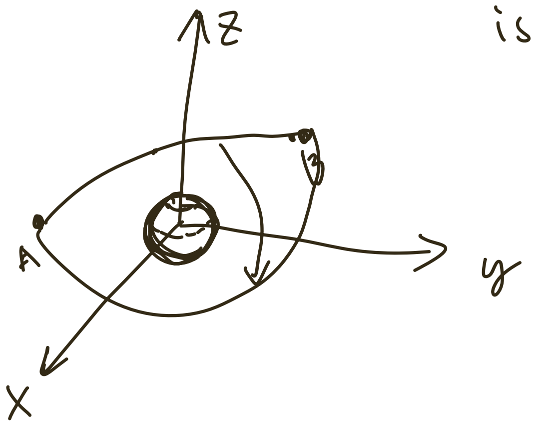


simply-connected region ✓

regions that are not simply-connected ✓ ✓

For R^3, $\{ (x, y, z) \in R^3 ; (x, y, z) \neq \boxed{x^2+y^2+z^2 \leq 1} \}$

is still a simply-connected region ✓



Curl and Divergence

Definition

The *curl* and *divergence* of a 3-dimensional vector field $F = (f_1, f_2, f_3)$ are defined as

$$\longrightarrow \quad \operatorname{curl} F = \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix},$$

$$\longrightarrow \quad \operatorname{div} F = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3.$$

Example: $\mathbf{F}(x, y, z) = \underbrace{y^2 z^3}_{f_1} \mathbf{i} + \underbrace{2xyz^3}_{f_2} \mathbf{j} + \underbrace{3xy^2z^2}_{f_3} \mathbf{k}$

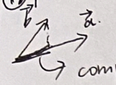
$\operatorname{Curl} F = 0$

$\operatorname{div} F = 2xz^3 + 6xy^2z$

$$\operatorname{Curl} F = \begin{pmatrix} 6xyz^2 - 6xyz^2 \\ \cdots \\ \cdots \end{pmatrix}$$

Yu Jiarui :)

(12节)



$\text{proj}_{\vec a}\vec b = \dfrac{\vec a\cdot\vec b}{|\vec a|^2}\vec a$

$\text{comp}_{\vec a}\vec b = \dfrac{\vec a\cdot\vec b}{|\vec a|}$

$$\begin{cases}\vec a\cdot(\vec b\times\vec c) = (\vec a\times\vec b)\cdot\vec c \ (\text{体积}) \\ \vec a\times(\vec b\times\vec c) = (\vec a\cdot\vec c)\vec b-(\vec a\cdot\vec b)\vec c\end{cases}$$

- Equations of Lines and Planes

L: $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

Line segment: $\vec r(t) = (1-t)\vec r_0 + t\vec r_1$ $(0\le t\le 1)$ 为2个端点

P: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\vec n = \langle a,b,c\rangle$ is normal vector

- 求两面交线: $\begin{cases}\text{step 1: 找点} \\ \text{step 2: } \vec v = \vec n_1\times\vec n_2\end{cases}$
- Distance: $D = \dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$
- Quadric

① $\vec v = A^{-1}(-b)$

$K = b^T v + c$

Ellipsoid: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

Elliptic Paraboloid: $\dfrac{z}{c} = \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$

Hyperbolic Paraboloid: $\dfrac{z}{c} = \dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$

Cone: $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$

Hyperboloid of One sheet: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

two sheets: $-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

$\displaystyle\int_{-\pi}^{\pi}\frac{d\theta}{(2+\cos\theta+\sin\theta)^2}$ subs $t=\tan(\frac{\theta}{2})$

$\displaystyle = \int_{-\infty}^{+\infty}\frac{\frac{2}{1+t^2}}{\left(2+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}\right)^2}\,dt$

$= \sqrt2\pi$

- line integral in space.

$\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\cdot dt$

- In Vector field:

$\displaystyle W = \int_a^b F(\vec r(t))\cdot\vec r'(t)\cdot dt$

$\dfrac{|x|+|y|}{\sqrt2} \le \sqrt{x^2+y^2} \le z \le \dfrac{1+|x|+|y|}{2}$

so $|x|, |y| \le 1+\sqrt2$

追踪它: 先固定一个坐标不动, 观察其他2个

(13节) Vector Function and Space Curve.

- A vector function $\vec r$ is continuous at a if $\lim_{t\to a}\vec r(t) = \vec r(a)$
- $\dfrac{d\vec r}{dt} = \vec r'(t) = \lim_{h\to0}\dfrac{\vec r(t+h)-\vec r(t)}{h}$
- $\vec T(t) = \dfrac{\vec r'(t)}{|\vec r'(t)|}$ ($\vec r'(t)$ exists, $\vec r'(t)\ne0$)
- arclength: $L = \int_a^b|\vec r'(t)|\,dt$
- $\vec N(t) = \dfrac{\vec T'(t)}{|\vec T'(t)|}$ $\vec B(t) = \vec T(t)\times\vec N(t)$
- curvature:

$k = \dfrac{|\vec T'(t)|}{|\vec r'(t)|}$

$k = \dfrac{|\vec r'(t)\times\vec r''(t)|}{|\vec r'(t)|^3}$

for $y=f(x)$, $k = \dfrac{|f''(x)|}{[1+(f'(x))^2]^{\frac32}}$

- osculating circle:

$R = \dfrac1k$

(q) center: $f(\vec t_0) + R\cdot\vec N(t_0)$

Example: $C(t) = q - 54\cos t\,\vec N(2) + 54\sin t\,\vec T(1)$

- Frenet-Serret formulas:

1. $d\vec T/ds = k\vec N$
2. $d\vec N/ds = -k\vec T + \tau\vec B$
3. $d\vec B/ds = -\tau\vec N$
4. $\tau = \dfrac{(\vec r'\times\vec r'')\cdot\vec r'''}{|\vec r'\times\vec r''|^2}$

- Line Integral

Definition: A differential 1-form $w: D\to(R^n)^*$, $D\subseteq R^n$, is said to be exact if there exists a function $f: D\to R^n$ satisfying $w=df$ (a so-called antiderivative). Equivalently, the vector field F corresponding to w is a gradient field, $F = \nabla f$

- If $w: D\to(R^n)^*$, $D\subseteq R^n$, is exact then D must be open, because $w(x) = df(x)$ for $x\in D$ imply in particular $D = D^\circ$

Q5. show that K is compact.

K is closed, because it's defined by weak inequalities $g(x,y,z)\le C$ involving one or more continuous functions g.

For bounded... 计算一端

$\int_C\vec F\cdot d\vec r = \int_C P\,dx+Q\,dy+R\,dz$ where $\vec F = P\hat i+Q\hat j+R\hat k$

Q1 (的 f) replace $y-2x$ by $(y+2x)$

$\int_C\nabla f\cdot d\vec r = f(\vec r(b)) - f(\vec r(a))$

(14):

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if for every number $\varepsilon>0$ there is a corresponding number $\delta>0$ such that if $(x,y)\in D$, and $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$ then $|f(x,y)-L|<\varepsilon$

- Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a,b). If the function f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$
- Laplace's equation

$\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}=0$

- wave equation

$\dfrac{\partial^2u}{\partial t^2} = a^2\dfrac{\partial^2u}{\partial x^2}$

- Differentials.

$dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$.

Δz 是沿切线变化

- Implicit.

$\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z}$

- Gradient Vector

$\nabla f(x,y) = \langle f_x, f_y\rangle$

$D_uf(x,y) = \nabla f(x,y)\cdot\vec u$ = slope

- Hess. $= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix}$ $>0 \Rightarrow$ local min/max; <0 saddle point.

local min/max: $\nabla f(x,y) = (0,0)$ (no 限制时)

- global max/min: $\begin{cases}\text{① all critical point} \\ \text{② boundary.}\end{cases}$
- Lagrange multiplier

for $\nabla f(x_0,y_0,z_0) = \lambda\nabla g(x_0,y_0,z_0)$

$\nabla g\ne0$

For $\nabla f(x_0,y_0,z_0) = \lambda\nabla g + \mu\nabla h$

$\nabla g\ne0$, $\nabla h\ne0$, ∇g and ∇h linear independent

(15) if $m\le f(x,y)\le M$ for all (x,y) in D, then $mA(D)\le\iint_D f(x,y)\,dA\le MA(D)$

- Moment

$M_x = \iint_D y\,\rho(x,y)\,dA$

$\bar x = \dfrac{M_x}{m}$, $m = \iint_D\rho(x,y)\,dA$

- Moment of Inertia

$I_x = \iint_D y^2 \rho(x,y)\, dA$

- Surface.

$A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \cdot dA$

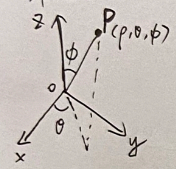
- triple. $V_E = \iiint_E dV = \iint_D z\, dA$

$m = \iiint_E \rho(x,y,z)\, dV$

$M_{yz} = \iiint_E x\rho(x,y,z)\, dV$

$\bar{x} = \frac{M_{yz}}{m}$

- Spherical. $+ \ \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$



- change of Variable.

$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$d^3(x,y,z) = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$

16 极值存在定理：S is compact, f is continuous ⇒ f 在S上有极大/小值

- For $H_f(\vec{x_0}) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

半正定：$A>0$ 且 $AC-B^2 = 0$

正定：$A>0$ 且 $AC-B^2>0$

负定：$A<0$ 且 $AC-B^2>0$

indefinite：$AC-B^2<0$

- Conformal：$J_f^T(x) \cdot J_f(x) = \lambda I,\ \lambda \neq 0$

- Differentiable：

Let $f: D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, be a function with coordinate functions $f_1, \cdots f_m$ and $\mathbf{x} \in D^\circ$.

① If f is differentiable at $\mathbf{x}$ then f is continuous at $\mathbf{x}$.

② If all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists near $\mathbf{x}$, then f is differentiable at $\mathbf{x}$

$\boxed{Q_3}$ $\boxed{b)}$ Show that there's a point on S minimized distance. S: $xyz - x - y - z = 2$

Consider a point on S: $P(-1,-1,-1)$

The length function $(x,y,z) \to |(x,y,z)|$ is continuous and attains a minimum d_0 on the set $C = \{(x,y,z) \in S;\ x^2+y^2+z^2 \le 3\}$ which is closed, bounded and non-empty. Since $d_0 \le \sqrt{3}$ and all points in $S \setminus C$ have length $> \sqrt{3}$, the point $P_0 = (x_0, y_0, z_0)$ has the required property.

- Monotone Convergence Theorem

If ① $f_k(x) \le f_{k+1}(x)$ (for all $x \in \mathbb{R}^n$ 与 for all $k \in \mathbb{N}$)

and ② $(\int f_k)_{k\in\mathbb{N}}$ is bounded

Then：$\int \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int f_k$

- Bounded Convergence Theorem

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of integrable functions on $\mathbb{R}^n$ converging almost everywhere and that there exists an integrable function $\phi \ge 0$ on $\mathbb{R}^n$ such that $|f_k(x)| \le \phi(x)$ for all $x \in \mathbb{R}^n$. Then the limit function $f(x) = \lim_{k\to\infty} f_k(x)$ is integrable with

$\int f = \lim_{k\to\infty} \int f_k$

- 考虑 $F(x) = \int_{y \in Y} f(x,y)\,dy$

如果对于每一个在定义域内的 x，$f(x,y)$ 都可积，那么 $F(x)$ is well-defined.

- If $x \to f(x,y)$ is continuous for each $y \in Y$ and there exists an integrable function $\phi: Y \to \mathbb{R}$ such that $|f(x,y)| \le \phi(y)$ for all $(x,y) \in X \times Y$, then F is continuous

- If $x \to f(x,y)$ is a C^1-function for each $y \in Y$ and there exists an integrable function $\phi: Y \to \mathbb{R}$ such that

$\left|\frac{\partial f}{\partial x_i}(x,y)\right| \le \phi(y)$ for all $(x,y) \in X \times Y$

then F is a C^1-function and

$\frac{\partial F}{\partial x_i}(x) = \int_Y \frac{\partial f}{\partial x_i}(x,y)\, d^n y$.

Q_6. show that $F'(x)$ exists and 求导积分可换序：

$F(x) = \int_0^\infty \frac{\sin t}{t} e^{-xt} dt$, $x \in [0, +\infty)$

$|f_x(x,t)| = |-\sin t\, e^{-xt}| \le e^{-xt} \le \phi(t)$

choose $\phi(t) = e^{-\frac{x}{2}t}$

and $\phi(t) = e^{-\frac{x}{2}t}$ is independent of x and integrable over $[0, +\infty)$, so the function F can be differentiated under the integral sign

Q_1. (c) False. For the closed curve

$\gamma(t) = (2\cos t, 2\sin t)$, $t \in [0, 2\pi]$, we have $\int_\gamma \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi \neq 0$, and hence $F(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$, $(x,y) \in D$, is not a gradient field. But F satisfies $(f_1)_y = (f_2)_x$

Q_2. $g_x = 0$ 且 $(1,1)$ 处 have 4 critical points.

since Δ is closed and bounded and g is continuous, g attains both a maximum and a minimum on Δ. One of these must be in the interior Δ° and hence yield a 4th critical point.

- Smooth：$\vec{r}(t)$ is smooth if on an interval I if $\vec{r}'$ is continuous and $\vec{r}'(t) \neq 0$ on I.

-

① continuous：If $x \to f(x,y)$ is continuous for each $y \in Y$ and there exists an integrable function $\phi: Y \to \mathbb{R}$ such that $|f(x,y)| \le \phi(y)$ for all $(x,y) \in X \times Y$, then F is continuous

$F = \int_Y f(x,y)\, d^n y$

- Fundamental：(continuous and differentiable)

$F(x) = F(0) + \int_0^x F'(t)\,dt$

- $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

- exact ① 能求 antiderivative ⇔ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

② 定义域 simply-connected, open. 不然世界不可能

- $\operatorname{curl}(F) = 0$ ⇔ locally exact.

- compact (开放式) { closed：每个条件的限制条件. bounded：需计算，有界

- 有个极值的条件：一个连续且 compact 的函数

- Curl：$F = P\hat{i} + Q\hat{j} + R\hat{k}$

$\operatorname{curl} F = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}$ (vector)

- $\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ (scalar)

- Theorem.

① For any C^2-function $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, we have $\operatorname{curl}(\nabla f) = 0$

② For any C^2 vector-field $G = (g_1, g_2, g_3): D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$, we have $\operatorname{div}(\operatorname{curl} G) = 0$

- winding Form：$w = \frac{x\,dy - y\,dx}{x^2+y^2}$

绕一圈 $\int_\gamma w = 2\pi$

- 连续：$\lim_{x\to c} f(x) - f(c) = 0$

一致连续：$\lim_{x_1 \to x_2} (f(x_1) - f(x_2)) = 0$

③ If $x \to f(x,y)$ is a C^1-function for each $y \in Y$ and there exists an integrable function $\phi: Y \to \mathbb{R}$ such that $\left|\frac{\partial f}{\partial x_i}(x,y)\right| \le \phi(y)$ for all $(x,y) \in X \times Y$ and $1 \le i \le m$, then F is itself a C^1-function.