单调收敛定理

定理1：

Theorem (Monotone Convergence Theorem, B. LEVI)

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a non-decreasing sequence (i.e., $f_k(\mathbf{x}) \le f_{k+1}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $k \in \mathbb{N}$) of integrable functions on $\mathbb{R}^n$ and that the sequence of integrals $(\int f_k)_{k\in\mathbb{N}}$ is bounded. Then $f(\mathbf{x}) = \lim_{k\to\infty} f_k(\mathbf{x})$ is finite almost everywhere, and the limit function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is integrable with

$$② \int f = \lim_{k\to\infty} \int f_k.$$

条件：① f_k 单调递增

② f_k 可积.

③ $(\int f_k)_{k\in N}$ 有界

$\exists M$. s.t. $\forall k\in N$. $\int_R f_k(x)dx \le M$

结论：① $\exists f$. $f(x) = \lim_{k\to\infty} f_k(x)$.

Corollary ("Improper" Lebesgue Integration)

Suppose $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is a nested sequence of measurable sets in $\mathbb{R}^n$, $A = \bigcup_{k=1}^{\infty} A_k$, and $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, is a function defined on A (i.e., $D \supseteq A$). Then f is integrable over A if and only if f is integrable over each A_k and the sequence $\left(\int_{A_k} |f|\right)_{k\in\mathbb{N}}$ is bounded (from above). If this is the case, we have

$$\int_A f = \lim_{k\to\infty} \int_{A_k} f.$$

9. $\lim_{k\to\infty} \left(\int_0^1 x - x^k \cos x\,dx\right)$ is equal to

☐ 1 ☐ $\frac{1}{4}$ ☐ 0 ☐ $\frac{1}{2}$ ☐ $\frac{\pi}{6}$

4. $\lim_{k\to\infty} \int_0^1 \frac{t(1+k^2t^2)}{(1+kt)^2}\,dt$ is equal to

☐ 0 ☐ $\frac{1}{3}$ ☐ $\frac{1}{2}$ ☐ 1 ☐ $+\infty$

定理2：

Theorem (Bounded Convergence Theorem, H. LEBESGUE)

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of integrable functions on $\mathbb{R}^n$ converging almost everywhere and that there exists an integrable function (integrable "bound") $\Phi \ge 0$ on $\mathbb{R}^n$ such that $|f_k(\mathbf{x})| \le \Phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Then the limit function $f(\mathbf{x}) = \lim_{k\to\infty} f_k(\mathbf{x})$ (extended to $\mathbb{R}^n$ in any way) is integrable with

$$\boxed{\int f = \lim_{k\to\infty} \int f_k.}$$

Notes

条件：① 存在 $\exists f$. $f(x) = \lim_{k\to\infty} f_k(x)$. ✓

② $|f_k(x)| \le \phi(x)$ ✓

ϕ 可积.

结论：① $\int \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int f_k$.

$f_k(x) = x - x^k \cos x$ $\quad k\to\infty$

9. $\lim_{k\to\infty} \left(\int_0^1 x - x^k \cos x\,dx\right)$ is equal to

☐ 1 ☐ $\frac{1}{4}$ ☐ 0 ☑ $\frac{1}{2}$ ☐ $\frac{\pi}{6}$

① $x\in[0,1]$, $k\to\infty$ $\quad x - x^k\cos x \to x$. limit function $f(x) = x$ ✓

② $|x - x^k\cos x| \le x + |x^k \cdot \cos x| \le x + x^k \le x+1$ $\quad \phi(x)$ $\quad \phi(x)$.

$\lim_{k\to\infty} \int_0^1 x - x^k\cos x\,dx = \int_0^1 x = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$.

$\int_0^1 f(x)dx$

做这类题的时候
要看它是不是bounded

$k \to \infty$

4. $\lim_{k\to\infty} \int_0^1 \frac{t(1+k^2t^2)}{(1+kt)^2}\,dt$ is equal to

☐ 0 ☐ $\frac{1}{3}$ ☑ $\frac{1}{2}$ ☐ 1 ☐ $+\infty$

$k \in \mathbb{N}$. $k > 0$.

① $x \in [0,1]$, $k \to \infty$. $\frac{t(1+k^2t^2)}{(1+kt)^2} = t\left(1 - \frac{2kt}{1+2kt+k^2t^2}\right)$

$= t\left(1 - \frac{2}{\frac{1}{kt} + kt + 2}\right)$

$k \to \infty$, $\frac{1}{kt} + kt \to \infty$

$\to t$. limit function, $f(t) = t$

② $\left| t\left(1 - \frac{2}{\boxed{\frac{1}{kt} + kt} + 2}\right)\right| \le t + t\frac{2}{\frac{1}{kt}+kt+2} \le \frac{3}{2}t$.

$\frac{1}{kt}+kt \ge 2$, $\frac{1}{kt}+kt+2 \ge 2$. $\frac{3}{2}t = \phi(t)$.

$\lim_{k\to\infty} \int_0^1 \frac{t(1+k^2t^2)}{(1+kt)^2}\,dt = \int_0^1 t\,dt = \frac{1}{2}t^2 = \frac{1}{2}$.

Theorem (Bounded Convergence Theorem, H. LEBESGUE)

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of integrable functions on $\mathbb{R}^n$ converging almost everywhere and that there exists an integrable function (integrable "bound") $\Phi \ge 0$ on $\mathbb{R}^n$ such that $|f_k(\mathbf{x})| \le \Phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Then the limit function $f(\mathbf{x}) = \lim_{k\to\infty} f_k(\mathbf{x})$ (extended to $\mathbb{R}^n$ in any way) is integrable with

$$\int f = \lim_{k\to\infty} \int f_k.$$

Notes

⇓ application: Parameter Integral 参数积分

Theorem

Suppose $f\colon X \times Y \to \mathbb{R}$ ($X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$) is such that $\mathbf{y} \to f(\mathbf{x},\mathbf{y})$ is integrable for each $\mathbf{x} \in X$ and $F\colon X \to \mathbb{R}$ is defined by $F(\mathbf{x}) = \int_Y f(\mathbf{x},\mathbf{y})\,d^n\mathbf{y}$.

2条性质

1. *If $\mathbf{x} \to f(\mathbf{x},\mathbf{y})$ is continuous for each $\mathbf{y} \in Y$ and there exists an integrable function $\Phi\colon Y \to \mathbb{R}$ such that $|f(\mathbf{x},\mathbf{y})| \le \Phi(\mathbf{y})$ for all $(\mathbf{x},\mathbf{y}) \in X \times Y$, then F is continuous.*
2. *If $\mathbf{x} \to f(\mathbf{x},\mathbf{y})$ is a C^1-function for each $\mathbf{y} \in Y$ and there exists an integrable function $\Phi\colon Y \to \mathbb{R}$ such that $\left|\frac{\partial f}{\partial x_j}(\mathbf{x},\mathbf{y})\right| \le \Phi(\mathbf{y})$ for all $(\mathbf{x},\mathbf{y}) \in X \times Y$ and $1 \le j \le m$, then F is itself a C^1-function and satisfies*

$$\frac{\partial F}{\partial x_j}(\mathbf{x}) = \int_Y \frac{\partial f}{\partial x_j}(\mathbf{x},\mathbf{y})\,d^n\mathbf{y} \quad \text{for } 1 \le j \le m.$$

这个被一个 $\phi(y)$ bound 住即可

条件

① $f(x,y)$ 连续.

② $|f(x,y)| \le \phi(y)$

$\Rightarrow$ F 连续.

f C^1-function. ① $\frac{\partial f}{\partial x}$ 存在. ✓

② $\frac{\partial f}{\partial x}$ 连续. ✓

③ $\left|\frac{\partial f}{\partial x_j}(x,y)\right| \le \phi(y)$. ✓

满足3个条件

就可以把求导 从外面放到里面.

$$F(x)=\int_0^1 \frac{\ln(t^2-2t\cos x+1)}{t}\,dt$$

5. For $F(x)=\int_0^\infty \frac{\sin(xt)}{1+t^3}\,\mathrm{d}t$ the derivative $F'(0)$ is

☐ 0 ☐ $\int_0^\infty \frac{\mathrm{d}t}{1+t^3}$ ☐ $\int_0^\infty \frac{\mathrm{d}t}{(1+t^3)^2}$ ☐ undefined ☑ $\int_0^\infty \frac{t}{1+t^3}\,\mathrm{d}t$

$$f(x,t)=\frac{\sin(xt)}{1+t^3}. \qquad \frac{\partial f(x,t)}{\partial x}=\frac{t\,\cancel{\sin}^{\cos}(xt)}{1+t^3}$$

$$\left|\frac{\partial f(x,t)}{\partial x}\right| \le \underbrace{\frac{t}{1+t^3}}_{\phi(t)}$$

∴我们可以把导数符号从积分号外面放到里面；

$$F'(x)=\int_0^\infty \frac{d\left(\frac{\sin(xt)}{1+t^3}\right)}{dx}\cdot dt$$

$x=0$

$$F'(x)=\int_0^\infty \frac{t\cos(xt)}{1+t^3}\cdot dt$$

10. For $F(x)=\int_0^1 \mathrm{e}^{xt^2}\,\mathrm{d}t$ the derivative $F'(1)$ is equal to

☐ e^{t^2} ☐ $\int_0^1 \mathrm{e}^{t^2}\,\mathrm{d}t$ ☐ $\int_0^1 t\mathrm{e}^{t^2}\,\mathrm{d}t$ ☐ $2\int_0^1 t\mathrm{e}^{t^2}\,\mathrm{d}t$ ☑ $\int_0^1 t^2\mathrm{e}^{t^2}\,\mathrm{d}t$

$$f(x,t)=e^{xt^2} \qquad \frac{\partial f(x,t)}{\partial x}=t^2e^{xt^2}. \qquad |t^2e^{xt^2}| \le t^2e^{t^2}.$$

$$F'(x)=\int_0^1 \frac{\partial f(x,t)}{\partial x}\,dt=\int_0^1 t^2e^{xt^2}\,dt$$

$x=1 \Rightarrow$ 选 e.

$$v^TA+b^T=0$$

$$v^TA=-b^T$$

$$(v^TA)^T=(-b^T)^T$$

$$A^Tv=-b$$

∵ A^T 是对称的矩阵

∴ $Av=-b$

$$v=A^{-1}(-b)$$