Next Wednesday, the last midterm.

形式和Mid 1 一样，10道选择，no cheatsheet，no calculator

☆ ① multiple integral

② Max / min / Saddle point

③ Quadratic form

☆ ④ Theorems of Lebusque integration

⑤ Slope / directional derivative / tangent plane

迈向二重积分 :)

$$\iint_D f(x, y)\, dA$$

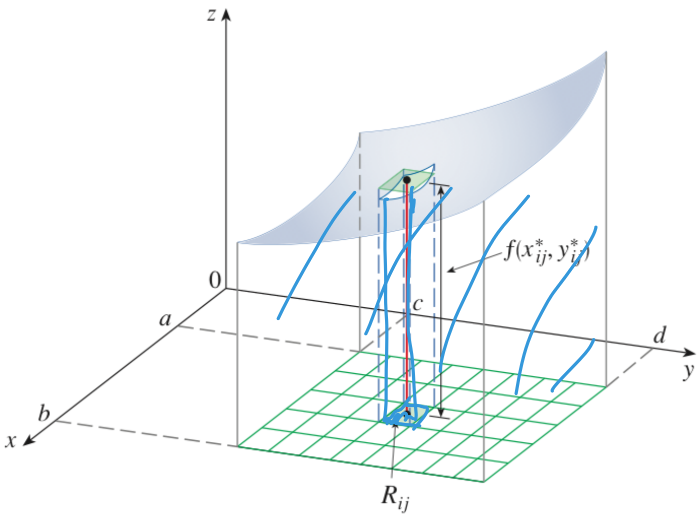


FIGURE 4

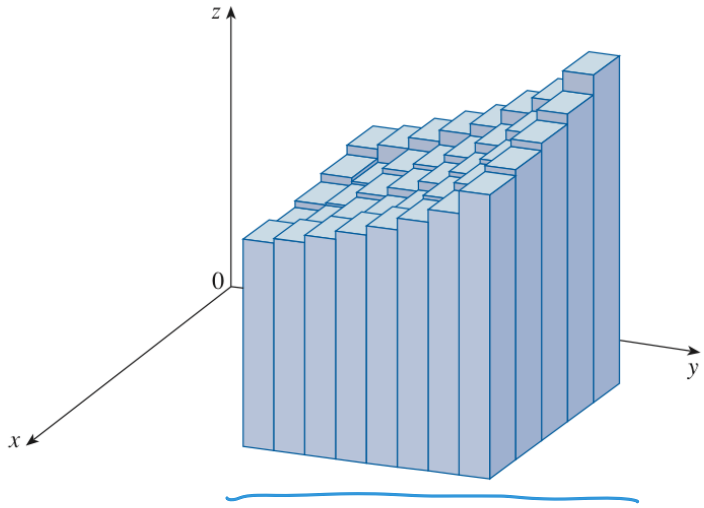


FIGURE 5

我们可以先积y再积x，也可以先积x再积y

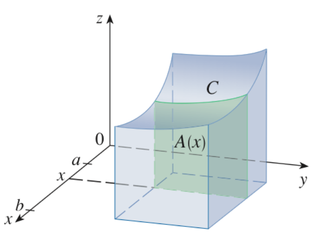


FIGURE 11

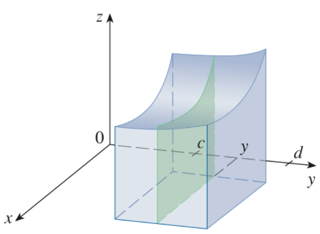


FIGURE 12

10 Fubini's Theorem If f is continuous on the rectangle

$$R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

then

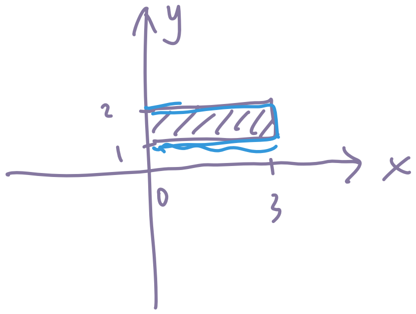
$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

$$\iint x^2 y^2\, dx\, dy = \int x^2\, dx \int y^2\, dy$$

$$\boxed{11} \quad \iint_R g(x)\, h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \quad \text{where } R = [a, b] \times [c, d]$$

1



$$\iint_D f(x, y)\, dA$$

Here $dA = dx\,dy$

EXAMPLE 4 Evaluate the iterated integrals.

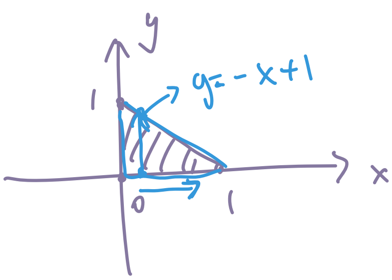
(a) $\int_0^3 \boxed{\int_1^2 x^2y\,dy}\,dx$ (b) $\int_1^2 \int_0^3 x^2y\,dx\,dy$

$\frac{27}{2}$

$= \frac{1}{2}x^2y^2\Big]_1^2 = 2x^2 - \frac{1}{2}x^2 = \frac{3}{2}x^2$

$\int_0^3 \frac{3}{2}x^2\,dx = \frac{1}{2}x^3\Big]_0^3 = \frac{27}{2}$

2



Example

Let Δ be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(0,1)$.
Compute the integral $\int_\Delta (y - x^2)\,d^2(x, y)$.

$\frac{1}{12}$

$\int_0^1 \boxed{\int_0^{-x+1} (y - x^2)\,dy}\,dx$

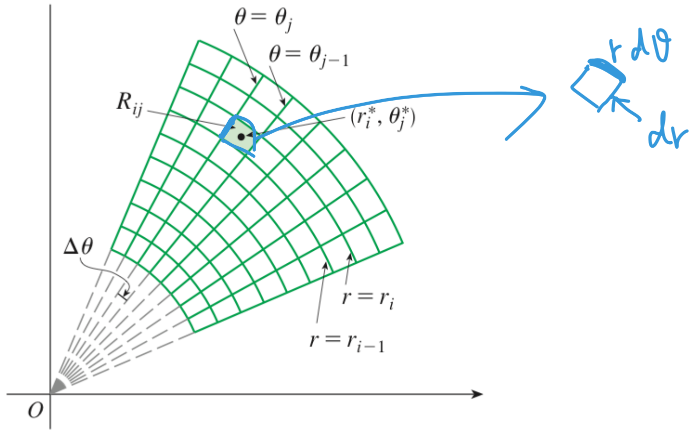
(x^2+1-2x) $\quad -x^2$

$= \frac{1}{2}y^2 - x^2y\Big]_0^{-x+1} = \frac{1}{2}(-x+1)^2 - x^2(-x+1) = x^3 - \frac{1}{2}x^2 - x + \frac{1}{2}$

$\int_0^1 x^3 - \frac{1}{2}x^2 - x + \frac{1}{2}\,dx = \frac{1}{4}x^4 - \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x\Big]_0^1 = \frac{1}{12}$

(3) 极坐标积分 :

$$\iint_D f(x, y)\, dA \qquad \text{Here } dA = r\,dr\,d\theta$$

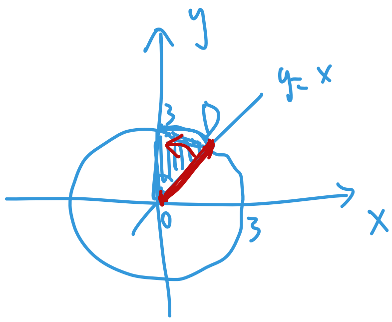


$(x, y) \longrightarrow (r, \theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

2. With $D = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \le 9,\ 0 \le x \le y\}$ the integral $\int_D y - x\, d^2(x,y)$ is equal to

- [ ] $9 + 9\sqrt{2}$
- [x] $-9 + 9\sqrt{2}$
- [ ] $9 - 9\sqrt{2}$
- [ ] $-9 - 9\sqrt{2}$
- [ ] 0



$r^2(\sin\theta - \cos\theta)$

$$\int_0^3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (r\sin\theta - r\cos\theta)\, r\, d\theta\, dr$$

$\Downarrow$

$$= \int_0^3 r^2\, dr \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta - \cos\theta\, d\theta$$

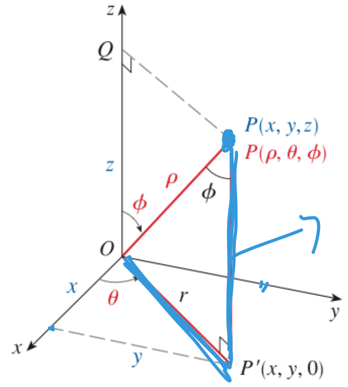
$$= \left.\frac{1}{3} r^3\right]_0^3 \cdot \Big[-\cos\theta - \sin\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 9\,(-1 + \sqrt{2})$$

迈向三重积分

$(x, y, z) \to (\rho, \phi, \theta)$

- 球坐标



$\rho \cdot \cos\phi$

FIGURE 5

Consequently, we have arrived at the following **formula for triple integration in spherical coordinates**.

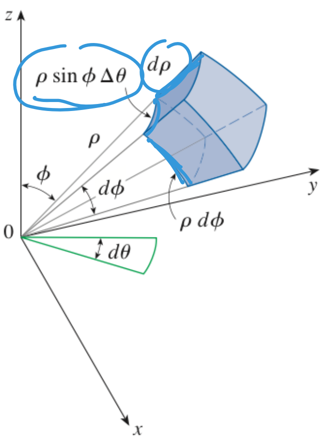


FIGURE 8
Volume element in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

3
$$\iiint_E f(x, y, z)\, dV$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \,\cos\theta, \rho \sin\phi \,\sin\theta, \rho\cos\phi)\, \boxed{\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$

dV

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$\boxed{x = \rho \sin\phi \,\cos\theta \qquad y = \rho \sin\phi \,\sin\theta \qquad z = \rho\cos\phi}$$

using the appropriate limits of integration and replacing dV by $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$. This is illustrated in Figure 8.

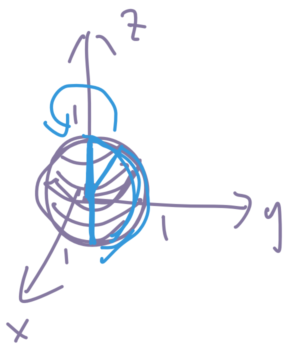
EXAMPLE 3 Evaluate $\iiint_B \boxed{e^{(x^2+y^2+z^2)^{3/2}}}\, dV$, where B is the unit ball:

ρ^2

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

$= \frac{4}{3}\pi(e - 1)$

$\rho \to (0, 1) \quad \phi \to (0, \pi) \quad \theta \to (0, 2\pi)$



$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3} \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$\Downarrow$

$$\int_0^1 e^{\rho^3} \cdot \rho^2 d\rho \cdot \int_0^\pi \sin\phi \, d\phi \cdot \int_0^{2\pi} d\theta$$

$$= \frac{1}{3} e^{\rho^3}\Big]_0^1 \cdot (-\cos\phi\Big]_0^\pi \cdot \Big[\theta\Big]_0^{2\pi}$$

$$= \left(\frac{1}{3}e - \frac{1}{3}\right) \times 2 \times 2\pi = 4\pi\left(\frac{1}{3}e - \frac{1}{3}\right)$$