ZJU-UIUC Institute
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Fall Semester 2021
Worksheet 11

Calculus III (Math 241)

W42 Do Exercises Exercises 25, 45, 49 in [Ste16], Ch. 14.8.
Which of the exercises in Homework 10 have a quick solution using the inequality between the arithmetic and geometric means from Exercise 49 ?

W43 Do the true-false quiz in [Ste16], p. 982.

W44 Determine the type of each quadric surface Q_a in the family

$$y^2 + xz + x - y - z = a, \quad a \in \mathbb{R}.$$

Hint: Q_a is central; the center can be found by rewriting the equation in a way similar to "completing the square".

W45 For $b > 1$ valuate $\displaystyle\int_1^b \frac{\mathrm{d}x}{x}$ without recourse to the Fundamental Theorem of Calculus.
Hint: Use upper and lower Darboux sums for partitions of $[1,b]$ that are in geometric progression; cf. lecture.

25. Consider the problem of minimizing the function $f(x, y) = x$ on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).
(a) Try using Lagrange multipliers to solve the problem.
(b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.
(c) Explain why Lagrange multipliers fail to find the minimum value in this case.

CAS **26.** (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum

41. Exercise
43. Exercis

44. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm² and whose total edge length is 200 cm.

45. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

46. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.

...the plane, and observe the elliptical

APPL

47. $f(x, y, z) = ye^{\cdots}$; $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = \ldots$

48. $f(x, y, z) = x + y + z$; $x^2 - y^2 = z$, $x^2 + z^2 = 4$

49. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.

(b) Deduce from part (a) that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

50. (a) Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $f_y(a, b) = \lim_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$

2. There exists a function f with continuous second-order partial derivatives such that $f_x(x, y) = x + y^2$ and $f_y(x, y) = x - y^2$.

3. $f_{xy} = \dfrac{\partial^2 f}{\partial x\, \partial y}$

4. $D_{\mathbf{k}} f(x, y, z) = f_z(x, y, z)$

5. If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b), then $\lim_{(x, y) \to (a, b)} f(x, y) = L$.

6. If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b).

7. If f has a local minimum at (a, b) and f is differentiable at (a, b), then $\nabla f(a, b) = \mathbf{0}$.

8. If f is a function, then

$$\lim_{(x, y) \to (2, 5)} f(x, y) = f(2, 5)$$

9. If $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$.

10. If $(2, 1)$ is a critical point of f and

$$f_{xx}(2, 1)\, f_{yy}(2, 1) < [f_{xy}(2, 1)]^2$$

then f has a saddle point at $(2, 1)$.

11. If $f(x, y) = \sin x + \sin y$, then $-\sqrt{2} \leqslant D_{\mathbf{u}} f(x, y) \leqslant \sqrt{2}$.

12. If $f(x, y)$ has two local maxima, then f must have a local minimum.

EXERCISES

Solutions

42 Ex. 25

(a) $f(x,y) = x$ has gradient $\nabla f(x,y) = (1,0)$, and $g(x,y) = y^2 + x^4 - x^3$ has gradient $\nabla g(x,y) = (4x^3 - 3x^2, 2y)$. The critical points of g are $(0,0)$ and $(3/4,0)$. Of these only $(0,0)$ is on the curve $y^2 + x^4 - x^3 = 0$. It follows that in all curve points $\neq (0,0)$ we can apply the theorem on Lagrange multipliers and conclude that a local extremum of f on the curve must satisfy

$$(1,0) = \lambda(4x^3 - 3x^2, 2y) \quad \text{for some } \lambda \in \mathbb{R}.$$

The second equation, $2\lambda y = 0$, forces $y = 0$, since $\lambda = 0$ is impossible, and further $x = 1$, since $y^2 + x^4 - x^3 = x^3(x-1) = 0$ and $(x,y) \neq (0,0)$. Thus the only solution is $(x,y) = (1,0)$ (and $\lambda = 1$). The point $(1,0)$, however, is not a minimum of f on the curve, since it has x-coordinate 1 whereas $(0,0)$ has x-coordinate 0. (In fact, the point $(1,0)$ is the unique maximum of f on the curve; see the plot below.)

(b) Since $y^2 + x^4 - x^3 = y^2 + x^3(x-1) = 0$ implies $0 \leq x \leq 1$, $f(0,0) = 0$ is the unique minimum of f on the curve; cf. also the plot below. Since $\nabla f(0,0) = (1,0)$, $\nabla g(0,0) = (0,0)$, the equation $\nabla f(0,0) = \lambda \nabla g(0,0)$ doesn't hold for any λ.

(c) Because the minimum value is attained at a critical point of g (visible as a cusp of the curve).

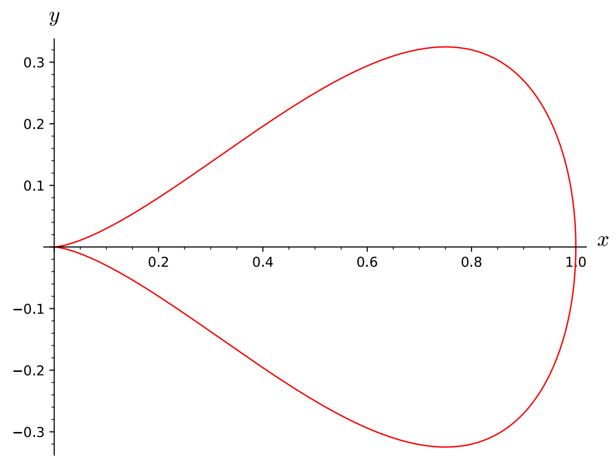


Figure 1: The curve $y^2 + x^4 - x^3 = 0$

Ex. 45 This exercise asks for the determination of the extrema of $f(x,y,z) = x^2 + y^2 + z^2$ under the constraints $g_1(x,y,z) = x + y + 2z - 2 = 0$ and $g_2(x,y,z) = x^2 + y^2 - z = 0$. Since $g = (g_1, g_2)$ has

$$\operatorname{rk} \mathbf{J}_g(x,y,z) = \operatorname{rk} \begin{pmatrix} 1 & 1 & 2 \\ 2x & 2y & -1 \end{pmatrix} = 2$$

except for $x = y = -\frac{1}{4}$, which leads to $g_1(-1/4, -1/4, z) = 2z - 5/2 = 0$, $g_2(-1/4, -1/4, z) = 1/8 - z = 0$ and has no solution, any extremum of f under the given constraints must satisfy the Lagrange multiplier condition

$$\nabla f(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2x & 2y & -1 \end{pmatrix}$$

and hence the system of equations

$$
\begin{aligned}
2x &= \lambda_1 + 2\lambda_2 x,\\
2y &= \lambda_1 + 2\lambda_2 y,\\
2z &= 2\lambda_1 - \lambda_2,\\
x+y+2z-2 &= 0,\\
x^2+y^2-z &= 0.
\end{aligned}
$$

Eliminating λ_1, λ_2 yields $x = y \vee z = -\frac{1}{2}$. The latter does not lead to a solution in view of $x^2+y^2 = z$. Hence solutions (x,y,z) must have $x = y$, from which it is easy to see that the solutions are $(-1,-1,2)$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. The point $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ (length $\frac{1}{2}\sqrt{3}$) is nearest to the origin, and $(-1,-1,2)$ (length $\sqrt{6}$) is farthest from the origin.

Ex. 49

(a) Since $x \mapsto \sqrt[n]{x}$ is strictly increasing, we may as well work with $p(x_1,\dots,x_n) = x_1x_2\cdots x_n$. (This simplifies computing the gradient.)

The maximum exists (provided that $c > 0$), since we can extend the feasible region $S = \{\mathbf{x} \in \mathbb{R}^n; x_i > 0, x_1+\cdots+x_n = c\}$ to $\overline{S} = \{\mathbf{x} \in \mathbb{R}^n; x_i \geq 0, x_1+\cdots+x_n = c\}$, which is non-empty, closed and bounded; since p is continuous on $\overline{S}$, it attains a maximum on $\overline{S}$ and, since p vanishes on $\overline{S} \setminus S$ and takes positive values on S, any such maximum must be in S.

The Lagrange multiplier condition is

$$
\begin{aligned}
\nabla p(x_1,\dots,x_n) &= (x_2\cdots x_n, x_1x_3\cdots x_n, \dots, x_1\cdots x_{n-1}) = x_1\cdots x_n(1/x_1,\dots,1/x_n)\\
&= \lambda(1,\dots,1).
\end{aligned}
$$

Since the gradient $\nabla g(x_1,\dots,x_n) = (1,\dots,1)$ is nonzero on S, any maximum $\mathbf{x}^*$ must satisfy the Lagrange multiplier condition. It follows that $x_1^* = x_2^* = \cdots = x_n^*$ and hence that $x^* = (c/n,\dots,c/n)$ is the unique maximum.

(b) Writing $c = x_1+\cdots+x_n$, we have from a) $x_1\cdots x_n \leq (c/n)^n$,

$$
\sqrt[n]{x_1\cdots x_n} \leq \frac{c}{n} = \frac{x_1+\cdots+x_n}{n};
$$

equality holds iff $x_1 = x_2 = \cdots = x_n$.

Finally we solve some of the exercises in Homework 11 using the inequality between the arithmetic and geometric means:

43 1. True. Setting $y = b+h$ turns this into the definition of $f_y(a,b)$.

2. False. Clairaut's Theorem applies and gives that f_{xy} and f_{yx} must be equal; but

$$
\begin{aligned}
f_{xy}(x,y) &= \frac{\partial}{\partial y}(x+y^2) = 2y,\\
f_{yx}(x,y) &= \frac{\partial}{\partial x}(x-y^2) = 1.
\end{aligned}
$$

Thus f_{xy} and f_{yx} coincide only on the line $y = 1/2$, which has no inner point; contradiction. (A pedantic set theorist might object that the function with empty domain provides an example, but we won't follow him.)

3. False, because $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ and $f_{yx} \neq f_{xy}$ in general.
4. True. Assuming $\mathbf{k} = (0,0,1)$, the directional derivative of f in direction $\mathbf{k}$ is just the partial derivative f_z.
5. False. We have seen that $f(x,y) = \frac{xy}{x^2+y^2}$, $(a,b) = (0,0)$ provides a counterexample.
6. False. One needs continuity of f_x and f_y in (a,b) to conclude that f is differentiable in (a,b).
7. True. This holds even under the assumption that f is only partially differentiable at (a,b); cf. the lecture.
8. False. If f is not continuous at $(2,5)$, it doesn't have the indicated property, and such functions f exist.
9. False; $\nabla f(x,y) = (0, 1/y)$ is a 2-dimensional vector.
10. True, as proved in the lecture.
11. True; $\nabla f(x,y) = (\cos x, \cos y)$, $|\nabla f(x,y)| = \sqrt{\cos^2 x + \cos^2 y} \leq \sqrt{2}$ and hence also $|\mathrm{D}_{\mathbf{u}} f(x,y)| = |\nabla f(x,y) \cdot \mathbf{u}| \leq \sqrt{2}$ for any unit vector $\mathbf{u} \in \mathbb{R}^2$.
12. False. The function $f(x,y) = x^4 + y^4 - 4xy + 1$ discussed as example in [Ste16] is a counterexample for the statement with maximum/minimum exchanged, and hence $g(x,y) = -f(x,y) = -x^4 - y^4 + 4xy - 1$ provides a counterexample.

44 A clever way to solve this exercise is as follows: The equation can rewritten as

$$\left(y - \frac{1}{2}\right)^2 + (x-1)(z+1) = a - \frac{3}{4}.$$

This shows that the quadric Q_a (the one with parameter a) is central with center $(x,y,z) = \left(1, \frac{1}{2}, -1\right)$ (independent of a) and translation-equivalent to $y^2 + xz = a - \frac{3}{4}$. Using a further variable change, viz. $x = x' + z'$, $z = x' - z'$, $y = y'$, the latter is transformed into $y^2 + x^2 - z^2 = a - \frac{3}{4}$.

$$\implies Q_a \text{ is a } \begin{cases} \text{cone} & \text{if } a = \frac{3}{4}, \\ \text{hyperboloid of one sheet} & \text{if } a > \frac{3}{4}, \\ \text{hyperboloid of two sheets} & \text{if } a < \frac{3}{4}. \end{cases}$$

Of course, the standard method, discussed in the lecture, can also be used. The equation defining Q_a is equivalent to

$$q_a(x,y,z) = y^2 + xz + x - y - z - a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^\mathsf{T} \underbrace{\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2 \underbrace{\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}}_{\mathbf{b}}^\mathsf{T} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - a = 0.$$

Since $\operatorname{rk} \mathbf{A} = 3$, Q_a is central with center $\mathbf{v} = (v_1, v_2, v_3)$ determined by

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The solution is $\mathbf{v} = \left(1, \frac{1}{2}, -1\right)$, so that Q_a is equivalent to the quadric with equation

$$y^2 + xz + q_a(\mathbf{v}) = y^2 + xz + \left(\tfrac{1}{2}\right)^2 + 1(-1) + 1 - \tfrac{1}{2} - (-1) - a = y^2 + xz + \tfrac{3}{4} - a = 0.$$

Then we use the algorithm for transforming $\mathbf{A}$ into Sylvester canonical form:

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \xrightarrow[C1=C1+C3]{R1=R1+R3} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \xrightarrow{R3=R3-\frac{1}{2}R1} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix} \xrightarrow{C3=C3-\frac{1}{2}C1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix}$$
$$\xrightarrow[C3=2C3]{R3=2R3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\Longrightarrow Q_a$ is equivalent to the quadric with equation $x^2 + y^2 - z^2 + \frac{3}{4} - a = 0$, which is the same as obtained above.

45 For $N \in \mathbb{N}$ consider the partition $P = \{1, q, q^2, \ldots, q^{N-1}, b\}$ of $[1,b]$ where $q := \sqrt[N]{b}$, i.e, for $1 \le i \le N$ the i-th subinterval of P is $[x_{i-1}, x_i] = [q^{i-1}, q^i]$. Since $x \mapsto 1/x$ attains its maximum in $[q^{i-1}, q^i]$ at q^{i-1} and its minimum in q^i, we have

$$\begin{aligned} \overline{S}(1/x;P) &= \sum_{i=1}^{N} \frac{q^i - q^{i-1}}{q^{i-1}} = N(q-1), \\ \underline{S}(1/x;P) &= \sum_{i=1}^{N} \frac{q^i - q^{i-1}}{q^i} = N(1 - 1/q), \\ \overline{S}(1/x;P) - \underline{S}(1/x;P) &= N(q - 1/q) \to 0 \quad \text{for } N \to \infty, \end{aligned}$$

since $\sqrt[N]{q} \to 1$. This shows that $x \mapsto 1/x$ is Riemann-integrable over $[1,b]$ with

$$\int_1^b \frac{\mathrm{d}x}{x} = \lim_{N\to\infty} N\left(\sqrt[N]{b} - 1\right) = \lim_{N\to\infty} \frac{b^{1/N} - 1}{1/N} = \frac{\mathrm{d}}{\mathrm{d}x} b^x \bigg|_{x=0} = \ln b.$$

Lec 31 Notes by TA LZH

Quadrics & Classification 二次曲面 & 分类

Quadratic form 二次型： 让它=0 就可以构造出一个二次曲面

$$q(x) = \underbrace{x^T A x}_{\text{代表二次项系数}} + \underbrace{2b^T x}_{\text{一次项系数}} + \underbrace{c}_{\text{常数}}$$

$x \in R^{n\times 1}$ $\quad A \in R^{n\times n}$ $\quad b \in R^{n\times 1}$ $\quad c \in R$

例： $n=3.$ $\quad x=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad b=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad c=0$：

$$q(x) = x^T A x = (x \ y \ z)\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (x, y, z)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= x^2 + y^2 + z^2$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \qquad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x^T A = (a_{11}x + a_{21}y + a_{31}z \quad a_{12}x + a_{22}y + a_{32}z \quad a_{13}x + a_{23}y + a_{33}z)$$

$$x^T A x = a_{11}x^2 + a_{21}xy + a_{31}xz + a_{12}xy + a_{22}y^2 + a_{32}zy + a_{13}xz + a_{23}yz + a_{33}z^2$$

$$= \underline{a_{11}}x^2 + \underline{a_{22}}y^2 + \underline{a_{33}}z^2 + \underline{(a_{21} + a_{12})}xy + (a_{31} + a_{13})xz + (a_{32} + a_{23})yz$$

a_{11}是x^2的系数 ......

$$A = \begin{pmatrix} x^2\text{系数} & xy\text{系数}/2 & xz\text{系数}/2 \\ xy\text{系数}/2 & y^2\text{系数} & yz\text{系数}/2 \\ xz\text{系数}/2 & yz\text{系数}/2 & z^2\text{系数} \end{pmatrix}$$

General Quadrics: 二次曲面:

$$x^T A x + 2b^T x + c = 0$$

例:

$$x^2+y^2+z^2+xy+xz+yz+x+y+z+1=0 \ :$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad c = 2.$$

"Degenerate" 退化情况

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad c = 2.$$

☆ $x+y+z+1=0. \Rightarrow$ 平面

"non-degenerate" $\Leftrightarrow \begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$ 可逆.

二次曲面分类:

理论基础:

Definition
Tow quadratic forms q_1 and q_2 on $\mathbb{R}^n$ are said to be *equivalent* (notation $q_1 \sim q_2$) if there exists an invertible matrix $\mathbf{S} \in \mathbb{R}^{n\times n}$ such that $q_2(\mathbf{x}) = q_1(\mathbf{Sx})$.

$\sim$: "形状一样".

Theorem (SYLVESTER's Inertia Theorem)
For every quadratic form q on $\mathbb{R}^n$ there are unique integers $r, s, t \geq 0$ with $r + s + t = n$ such that

$$q(x_1, \ldots, x_n) \sim x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2. \qquad \text{(S)}$$

Note that $t = n - r - s$, and $t \geq 0$ is equivalent to $r + s \leq n$.

结论:

(广义的一样)

任何一个非退化 Quadric 都与以下之一形状一样:

$x^2+y^2+z^2=1$ Ellipsoid

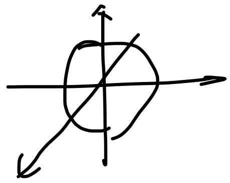
$x^2+y^2-z^2=1$ Hyperboloid of one sheet

$x^2-y^2-z^2=1$ Hyperboloid of Two sheets

$z = x^2+y^2$ Elliptic Paraboloid

$z = x^2-y^2$ Hyperbolic Paraboloid.

$x^2+y^2+z^2=1$



sphere

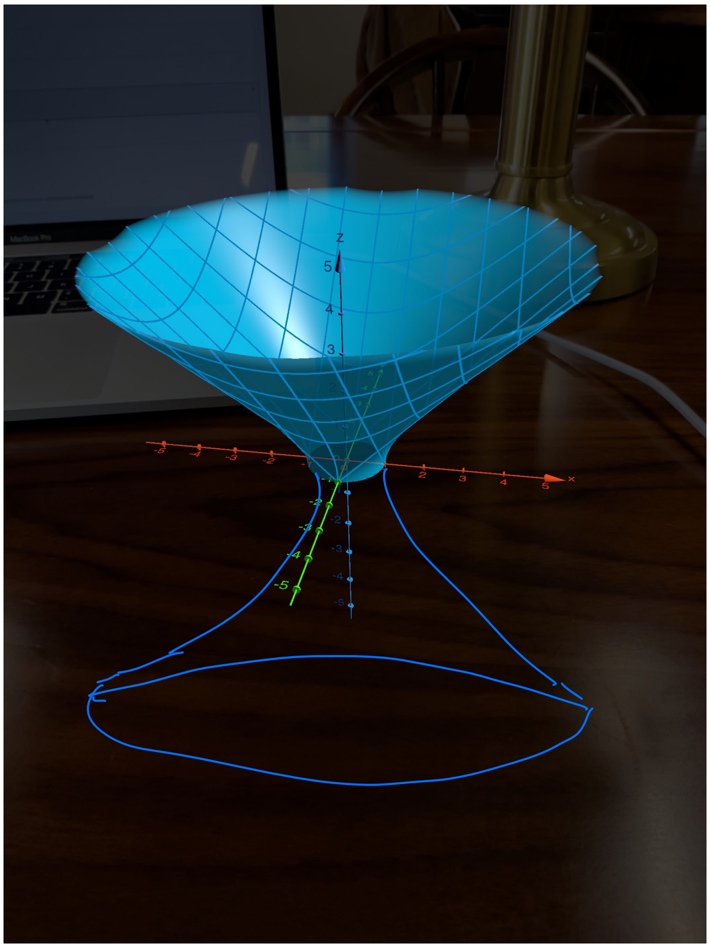
$x^2+y^2-z^2=1$

(xy平面以下和xy平面以上对称)

hyperboloid of 1 sheet

单叶双曲面

(连续的)



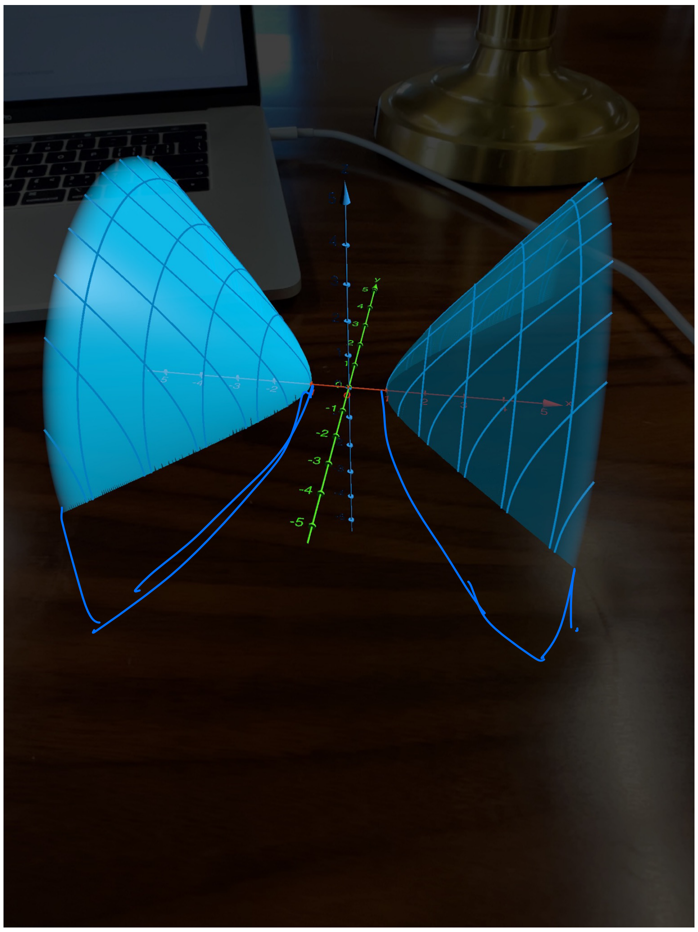
$x^2-y^2-z^2=1$

(xy平面以下和xy平面以上对称)

hyperboloid of 2 sheets

双页双曲面

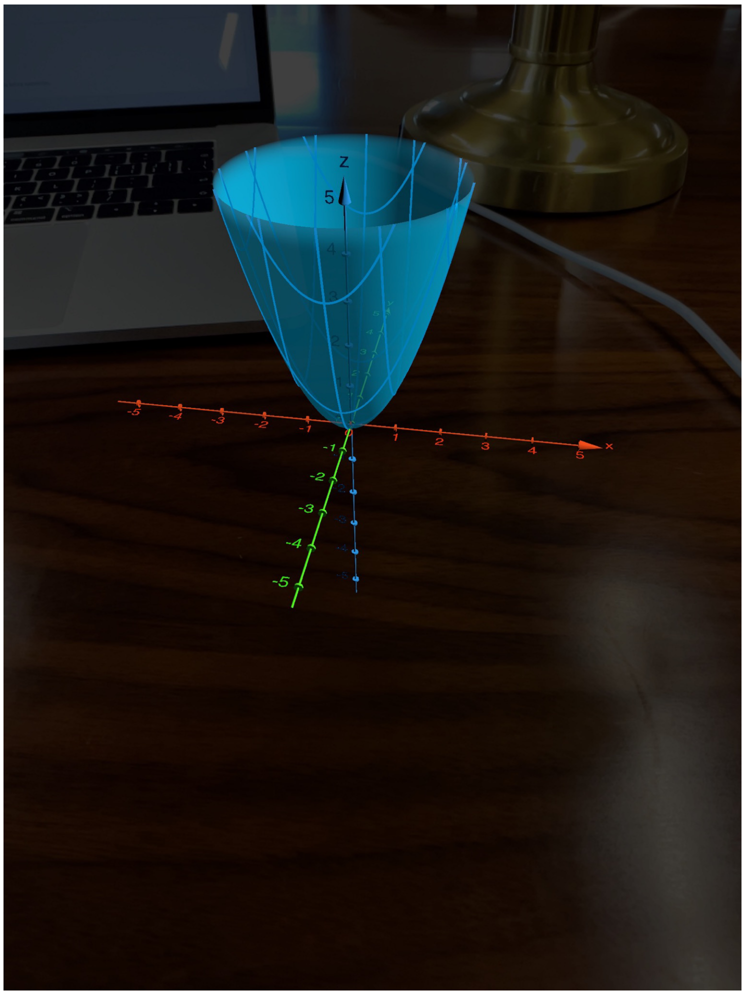
(2个连续体)



$z = x^2 + y^2$

elliptic paraboloid

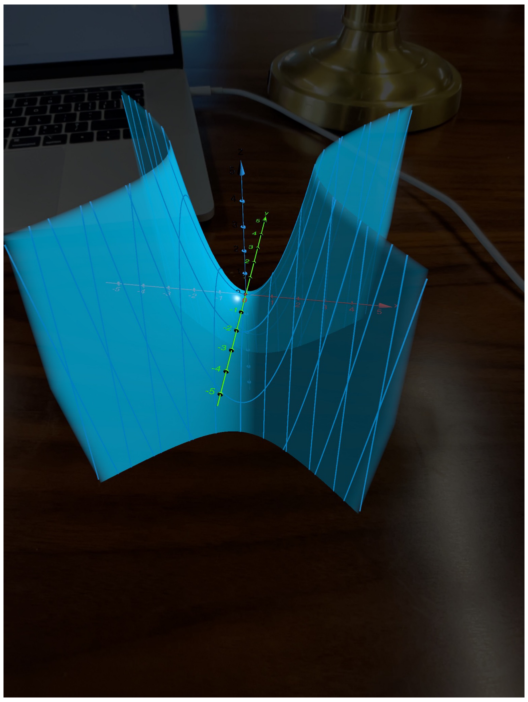
椭圆抛物面

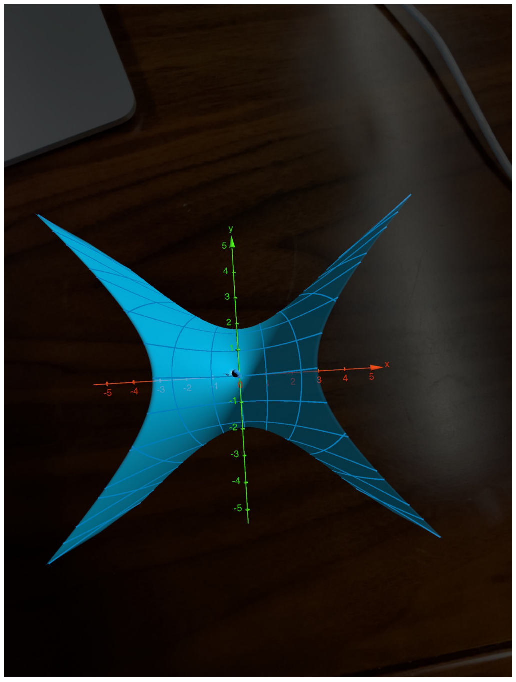


$z = x^2 - y^2$

hyperbolic paraboloid

双曲抛物面.





给定二次曲面的函数怎么判断形状？

$$x^TAx+2b^Tx+c=0$$ （先把A, B, C写出来）

希望对x进行变换消掉b： $x=Sx'+v$

其中 S满足 $S^TAS=\begin{pmatrix}\ddots\end{pmatrix}$ （只有对角线有元素）

$$(Sx'+v)^TA(Sx'+v)+2b^T(Sx'+v)+c=0$$

$$\Leftrightarrow\quad x'^T(S^TAS)x'+\underbrace{(2v^TAS+2b^TS)}_{=0}x'+\underbrace{v^TAv+2b^Tv+c}_{K}=0$$

$$2v^TAS+2b^TS=2\underbrace{(v^TA+b^T)}_{=0}S$$

$\Leftarrow$

$$v^TA+b^T=0\Leftrightarrow Av=-b\Leftrightarrow v=A^{-1}\cdot(-b)$$

解出v，代入

原二次曲面 $\Leftrightarrow$ $x'^T(S^TAS)x'+K=0$

对A高斯消元

$$\Leftrightarrow\quad ax^2+by^2+cz^2+K=0$$

A不满秩怎么办？

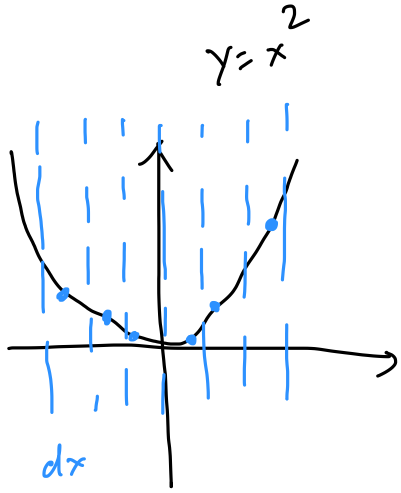
对A高斯消元，$\begin{pmatrix}\pm1&\\&\pm1\end{pmatrix}$ 同号则 $z=x^2+y^2$

异号则 $z=x^2-y^2$

Lec 32~34 Note by TA LZH

黎曼积分 vs 达布积分

Riemann Darboux



Definition (Riemann)

A function $f\colon [a,b] \to \mathbb{R}$ is *Riemann integrable* with $\int_a^b f(x)\,\mathrm{d}x = V$ if for every "error bound" $\epsilon > 0$ there is a "response" $\delta > 0$ such that for every partition $a = x_0 < x_1 < \cdots < x_N = b$ with subintervals $[x_{i-1}, x_i]$ of lengths $< \delta$ and any choice of "sample points" $x_i^* \in [x_{i-1}, x_i]$ we have

$$\left|\sum_{i=1}^{N} f(x_i^*)(x_i - x_{i-1}) - V\right| < \epsilon.$$

Darboux sums

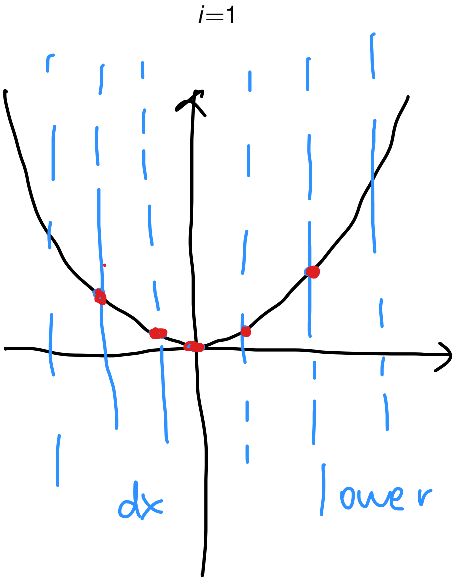
Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded (from above and from below). For a partition $P\colon a = x_0 < x_1 < \cdots < x_N = b$ and $i \in \{1, \dots, N\}$ let

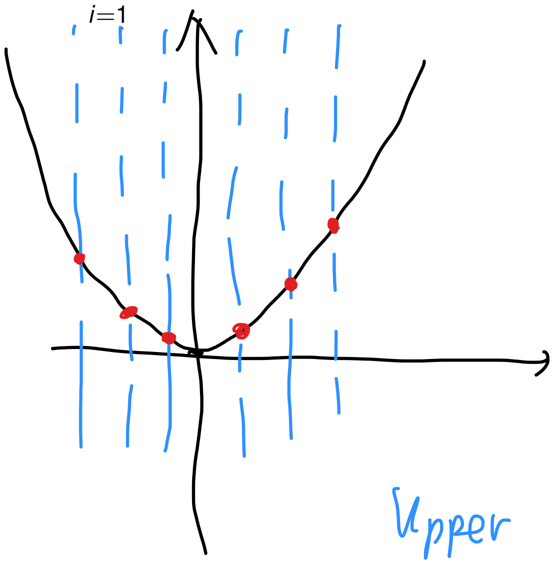
下确界 上确界

$$m_i = \inf\{f(x); x_{i-1} \le x \le x_i\}, \quad M_i = \sup\{f(x); x_{i-1} \le x \le x_i\},$$

and define the *lower* and *upper Darboux sum* of f with respect to P as

$$\underline{S}(P;f) = \sum_{i=1}^{N} m_i(x_i - x_{i-1}), \quad \overline{S}(P;f) = \sum_{i=1}^{N} M_i(x_i - x_{i-1})$$





这2种对于黎曼可积的函数其实对结果没影响，因为取的dx很小

Example

如果 $\underline{S} \neq \overline{S}$，就叫

Consider the *Dirichlet function* $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases} \quad \Rightarrow$$

(黎曼不可积)

勒贝格可积

Enumerating $\mathbb{Q}$ as $q_1, q_2, q_3, \ldots$ and setting $f_n(x) = 1$ if $x = q_n$ and $f_n(x) = 0$ otherwise (the domain of f_n is the same as for f), we have

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for every } x \in [0,1].$$

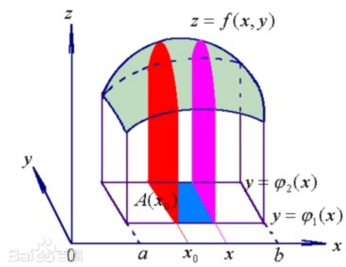
Clearly each f_n is Riemann integrable with $\int_0^1 f_n(x)\,\mathrm{d}x = 0$. In this situation we would like to have

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \left(\sum_{n=1}^{\infty} f_n(x) \right) \mathrm{d}x = \sum_{n=1}^{\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} 0 = 0$$

as well. But, unfortunately, f is not Riemann integrable. (Check that $\underline{S}(P;f) = 0$, $\overline{S}(P;f) = 1$ for every partition of $[0,1]$.)

迈向重积分

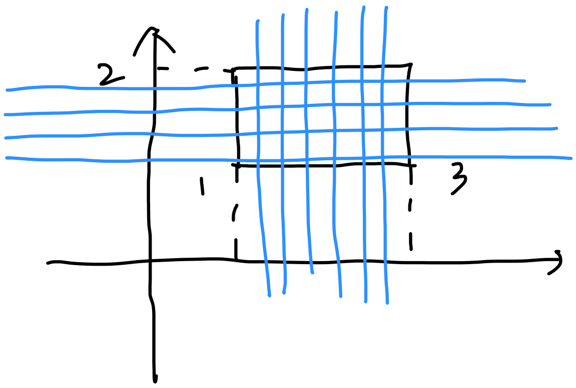
$\int_0^2 f(x)\,dx$



Theorem (Little Fubini)

Suppose $f\colon [a,b]\times[c,d]\to\mathbb{R}$, $(x,y)\mapsto f(x,y)$ is continuous. For $y\in[c,d]$ define $F(y)=\int_a^b f(x,y)\,\mathrm{d}x$. Then $F\colon[c,d]\to\mathbb{R}$ is Riemann integrable and satisfies

$$\int_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}^2(x,y)=\int_c^d F(y)\,\mathrm{d}y=\int_c^d\left(\int_a^b f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$



长方形铁板。

每处面积密度 xy 单位 质量/(长度^2)

求重量。

$$\int_{(1,3)\times(1,2)} xy\ d^2(x,y) = \iint xy\cdot dx\,dy$$

$$= \int_1^3\left(\int_1^2 xy\ dy\right)dx$$

$$= \int_1^3 \frac{1}{2}xy^2\Big|_1^2\ dx$$

$$= \int_1^3 \frac{3}{2}x\ dx$$

$$= \frac{3}{4}x^2\Big|_1^3$$

$$= \frac{27}{4}-\frac{3}{4} = 6.$$