Summary: 判断非退化二次曲面的形状

Step 1. 写出对应的 A, b, c

Step 2. 为了消掉一次项系数，令 $v^TA + b^T = 0 \iff v = A^{-1}\cdot(-b)$

Step 3. 解出常数 $k = v^TAv + 2b^Tv + c$

Step 4. 看 A

$\longrightarrow$ A 满秩：对 A 高斯消元. $\begin{pmatrix} \pm1 & & \\ & \pm1 & \\ & & \pm1 \end{pmatrix}$ $\begin{cases} x^2+y^2+z^2=1 \\ x^2+y^2-z^2=1 \\ x^2-y^2-z^2=1 \end{cases}$

$\searrow$ A 不满秩：对 A 高斯消元. $\begin{pmatrix} \pm1 & & \\ & \pm1 & \\ & & \cdot \end{pmatrix}$ 同号则 $z=x^2+y^2$；异号则 $z=x^2-y^2$

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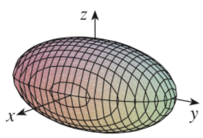
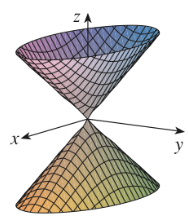
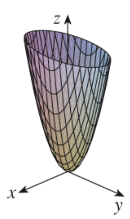
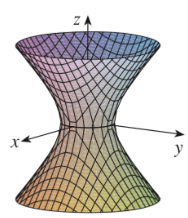
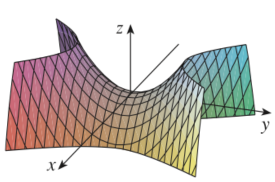
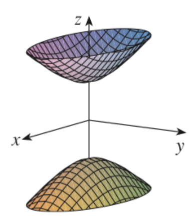
Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	**Cone** 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	**Hyperboloid of One Sheet** 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	**Hyperboloid of Two Sheets** 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

$2xy+2yz+2z-2=0$

8. The surface in $\mathbb{R}^3$ with equation $xy+yz+z=1$ is a

- [ ] ellipsoid
- [ ] hyperboloid of 1 sheet
- [ ] hyperboloid of 2 sheets
- [ ] elliptic paraboloid
- [x] hyperbolic paraboloid

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad b=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad c=-2$$

$$A \xrightarrow{R_1=R_1+R_2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3=R_3+R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[C_2=C_2-C_1]{C_3=C_3-C_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$2z^2-2xy+2x+2z=0$

4. The surface in $\mathbb{R}^3$ with equation $z^2-xy+x+z=0$ is a

- [ ] hyperbolic paraboloid
- [x] hyperboloid of 1 sheet
- [ ] cone
- [ ] hyperboloid of 2 sheets
- [ ] elliptic paraboloid

$$b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad c = 0$$

A^{-1}

$$\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

A^{-1}

$$v = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} =$$

$$k = v^T A v + 2b^T v + c = \underbrace{\begin{pmatrix} 0 & 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} - 1$$

$$\underline{k = -\frac{1}{2}}$$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{C_2 = C_2 - C_1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad -2x^2 + 2y^2 + 2z^2 = \frac{1}{2}$$

$$-x^2 + y^2 + z^2 - \frac{1}{2} = 0$$

$$-x^2+y^2+z^2=1$$