• Local maximum and minimum

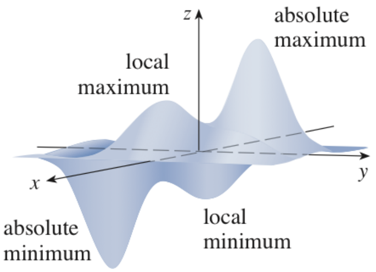


FIGURE 1

Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

• Critical point : 导数为0 或不存在

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f

• $\nabla f(x,y) = 0$ → local maximum
→ local minimum
→ saddle point

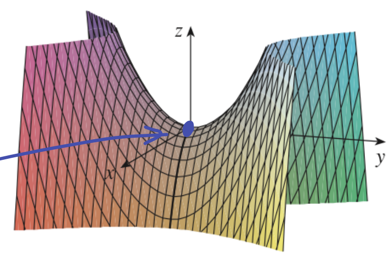


FIGURE 3
$z = y^2 - x^2$

用来
• 判断具体是个什么点：

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then (a, b) is a saddle point of f.

NOTE 1 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

- Global (Absolute) maximum and minimum

7 Definitio Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geqslant f(x, y)$ for all (x, y) in D.
- **absolute minimum** value of f on D if $f(a, b) \leqslant f(x, y)$ for all (x, y) in D.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

- 求极值：

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

- Lagrange Multipliers : Find the extreme values under some constraints.

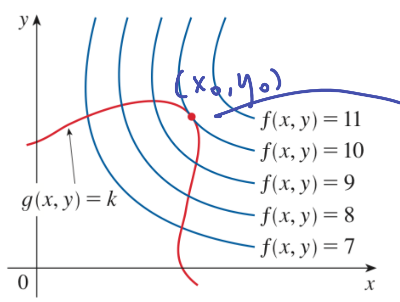


FIGURE 1

Example: 在 $g(x,y)=k$ 的限制下求 $f(x,y)$ 的极值

在取到极值的这个点，∇f 和 ∇g 平行

$\therefore \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ

Lagrange multiplier

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\left\{\begin{aligned} &\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ \text{and} \quad &g(x, y, z) = k \end{aligned}\right.$$

one constrain

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

For two constrains:

16

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)}$$

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

ZJU-UIUC Institute
Prof. Thomas Honold

Fall Semester 2023
Worksheet 9

Yu Jiarui

Calculus III (Math 241)

W25 Do Exercises 23, 34, 36 in [Ste21], Ch. 14.7.

W26 Do Exercise 5 in [Ste21], Ch. 14.8.

W27 Using Lagrange Multipliers, do Exercise 50 in [Ste21], Ch. 14.7.

23. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

$$\left.\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= 8y - 4x = 0 \end{aligned}\right\} \Rightarrow x = 2y$$

$f_{xx} = 2 \quad f_{yy} = 8$

$f_{xy} = -4$

$D = f_{xy}^2 - f_{xx} \cdot f_{yy} = 0$

33–40 Find the absolute maximum and minimum values of f on the set D.

33. $f(x, y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

34. $f(x, y) = x + y - xy$, D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$

35. $f(x, y) = x^2 + y^2 + x^2y + 4$, $D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$

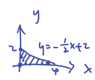
36. $f(x, y) = x^2 + xy + y^2 - 6y$, $D = \{(x, y) \mid -3 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 5\}$

34. let $\nabla f(x,y) = (1-y, 1-x) = 0$

$y = x = 1$

$f(1,1) = 1$

Check boundary:



$f(0,y) = y$, max $= 2$ min $= 0$

$f(x,0) = x$, max $= 4$ min $= 0$

$f(x, -\frac{1}{2}x+2) = \frac{1}{2}x + 2 - x(-\frac{1}{2}x+2)$

$= \frac{1}{2}x^2 - \frac{3}{2}x + 2 \quad x \in (0,4)$

$\frac{3}{2}$

min $= \frac{7}{8}$ max $= 4$

Thus, max $= 4$ min $= 0$

3–16 Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2$, $x^2 + y^2 = 1$

4. $f(x, y) = x^2y$, $x^2 + y^4 = 5$

5. $f(x, y) = xy$, $4x^2 + y^2 = 8$

5. $\nabla f = (y, x)$

$\nabla g = (8x, 2y)$

$$\begin{cases} \nabla f = \lambda \nabla g & , \nabla g \neq 0 \\ 4x^2 + y^2 = 8 \end{cases}$$

$$\begin{cases} y = 8x \cdot \lambda \\ x = 2y \cdot \lambda \\ 4x^2 + y^2 = 8 \end{cases} \Rightarrow$$

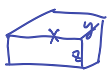
$\lambda = -4$	$\lambda = -4$	$\lambda = 4$	$\lambda = 4$
$x = -1$	$x = 1$	$x = -1$	$x = 1$
$y = 2$	$y = -2$	$y = -2$	$y = 2$

for $f(\pm 1, \pm 2)$, max $= 2$

min $= -2$

50. Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.

50.



let

$g = xyz = 1000$

$A = f(x,y,z) = 2xy + 2yz + 2xz$

$\nabla f = (2y + 2z, 2x + 2z, 2y + 2x)$

$\nabla g = (yz, xz, xy)$

$$\begin{cases} 2y + 2z = \lambda yz \\ 2x + 2z = \lambda xz \\ 2y + 2x = \lambda xy \\ xyz = 1000 \end{cases} \Rightarrow$$

$$\left.\begin{aligned} 2xy + 2zx &= \lambda xyz \\ 2xy + 2zy &= \lambda xyz \end{aligned}\right\} \Rightarrow x = y$$

Similarly, $x = y = z$

$\therefore x = y = z = 10$

$A = 600\ cm^2$