Anounncement

Mid 2 : Wed Nov 22 18:00 - 19:30

No calculator , Do not use your own sketch paper.

You can bring a double-sided hand-written cheatsheet of A4 paper.

Review session: Sat or Sun , Go over samples and relative topics.

A website that helps you better understand this class

Posted on: Monday, November 13, 2023 1:24:59 PM CST

We've developed an individual website to help you gain a deeper insight into this class, including notes, past samples and full lecture slides.

Hope you find this site useful for your future study.

URL: www.math241.top

Posted by: 韦沛然(Wei Peiran)
ZJU-UIUC Institute
Posted to: MATH241:Calculus III-1045-1046 (Fall 2023)

Note by YJR

Question 1 (ca. 11 marks)

Consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) \;=\; \frac{1}{x^2 - xy + y^2}.$$

Here $D \subseteq \mathbb{R}^2$ is the maximum possible domain for f.

a) Determine D.

b) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibilities $\pm\infty$), or show that the limit does not exist.

a) $x^2 - xy + y^2 \neq 0$

$(x - \frac{1}{2}y)^2 + \frac{3}{4}y^2 \neq 0$ The only solution for $(x-\frac{1}{2}y)^2 + \frac{3}{4}y^2 = 0$ is $\begin{cases} x=0 \\ y=0 \end{cases}$

Thus, $D = \mathbb{R}^2 / \{0,0\}$

this notation means exclude

b) Recall

— Ways to determine a limit exists:

① **1 Definitio** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$

② If the function is continuous at the given point, the limit at that point exists.

— Ways to determine a limit does not exist:

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

— For limit $= \pm\infty$ $\Rightarrow$ use polar coordinate.

In this question, $f(x,y) = \dfrac{1}{x^2 - xy + y^2} = \dfrac{1}{(x-\frac{1}{2}y)^2 + \frac{3}{4}y^2}$

by estimation, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \infty$

Thus, we intend to convert coordinate to polar coordinate.

let $x = r\cos\phi$
$y = r\sin\phi$

$f(x,y) = \dfrac{1}{r^2\cos^2\phi - r^2\sin\phi\cos\phi + r^2\sin\phi} = \dfrac{1}{r^2(1-\frac{1}{2}\sin(2\phi))}$

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

$\therefore$

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} \dfrac{1}{r^2(1-\frac{1}{2}\sin(2\phi))} = +\infty$

$\lim\limits_{|(x,y)|\to\infty} f(x,y) = \lim\limits_{r\to\infty} \dfrac{1}{r^2(1-\frac{1}{2}\sin(2\phi))} = 0$

$$f(x,y) = \frac{1}{x^2 - xy + y^2}.$$

c) Determine all critical points of f (if any).

Recall

- Critical point : $\begin{cases} f_x = f_y = 0 \\ \text{or} \\ \text{one of partial derivatives does not exist.} \end{cases}$

Here $\begin{cases} f_x = \dfrac{\partial f}{\partial x} = \dfrac{y-2x}{(x^2-xy+y^2)^2} \\ f_y = \dfrac{\partial f}{\partial y} = \dfrac{x-2y}{(x^2-xy+y^2)^2} \end{cases}$

$= \left((x-\frac{1}{2}y)^2 + \frac{3}{4}y^2\right)^2$

The only solution for $f_x = f_y = 0$ or DNE is $(x,y) = (0,0)$, which is not in the domain.

Thus, there's no critical points of f.

$$f(x,y) = \frac{1}{x^2 - xy + y^2}.$$

d) Determine the shape of the 1-contour C_1 of f and graph C_1 as accurately as possible (unit length at least 2 cm).

Hint: There are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $x^2 - xy + y^2 = \lambda_1 \left(\frac{x+y}{\sqrt{2}}\right)^2 + \lambda_2 \left(\frac{x-y}{\sqrt{2}}\right)^2$.

Recall

contour : $f(x,y) = k$, where k is a constant.

Here for 1-contour, means $k = 1$

$$\frac{1}{x^2 - xy + y^2} = 1 \quad \Rightarrow \quad x^2 - xy + y^2 = 1$$

By hint, we want to find λ_1 and λ_2

$$\lambda_1 \left(\frac{x+y}{\sqrt{2}}\right)^2 + \lambda_2 \left(\frac{x-y}{\sqrt{2}}\right)^2 = \frac{\lambda_1}{2}(x^2+y^2+2xy) + \frac{\lambda_2}{2}(x^2+y^2-2xy)$$

$$= \left(\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\right)x^2 + (\lambda_1 - \lambda_2)xy + \left(\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\right)y^2 = x^2 - xy + y^2$$

Thus, $\begin{cases} \frac{\lambda_1}{2} + \frac{\lambda_2}{2} = 1 \\ \lambda_1 - \lambda_2 = -1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{1}{2} \\ \lambda_2 = \frac{3}{2} \end{cases}$

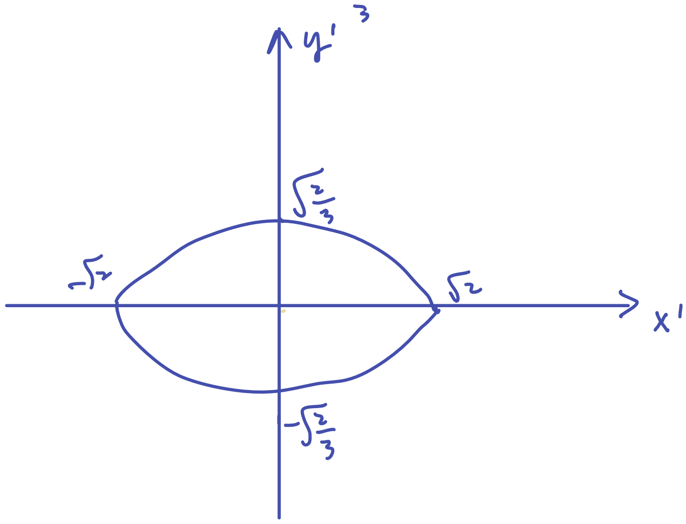
Thus, $\frac{1}{2}\left(\frac{x+y}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{x-y}{\sqrt{2}}\right)^2 = 1$ ← looks like an ellipse

let $x' = \frac{x+y}{\sqrt{2}} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$

$y' = \frac{x-y}{\sqrt{2}} = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y$

$$\Longleftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Sketch graph for $\frac{x'^2}{2} + \frac{y'^2}{\frac{2}{3}} = 1$



- examine this transformation

☆Recall

Rotation Matrix:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

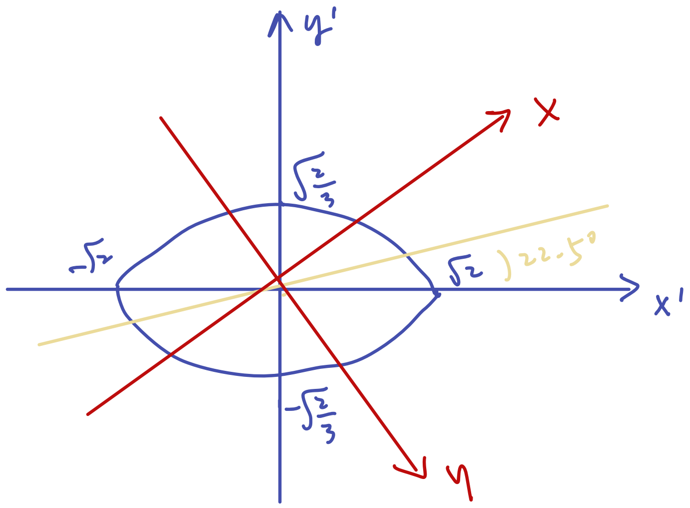
Rotate ϕ for counter-clockwise

Reflection Matrix:

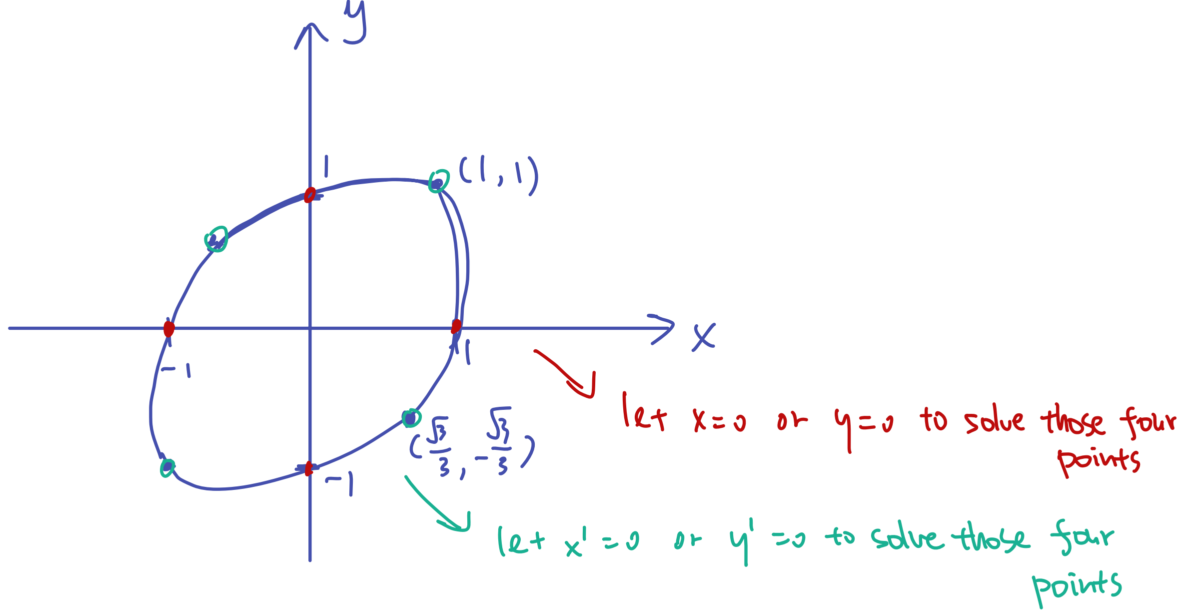
$$S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}, \quad \phi \in [0, 2\pi).$$

Reflection Axis: $\frac{\phi}{2}$ (in polar coordinate)

Here if we assume $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$ is a Reflection Matrix, $\phi = 22.5°$



$\Downarrow$ Resketch Graph C_1



e) Determine the slope of the graph G_f at $(1, 1)$ in the western (W) direction, and the maximal slope/direction of G_f at $(1, 1)$.

☆ Recall chapter 14.6 of textbook.

Slope = directional derivative = $D_u f(x,y)$, is a number

Gradient = $\nabla f(x,y)$, is a vector

Relation between them : $\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$

Direction vector
unit vector

Max. directional derivative : Along the direction of gradient

$$D_u f(x,y)_{max} = |\nabla f(x,y)|$$

Here we have
$$\begin{cases} f_x = \dfrac{\partial f}{\partial x} = \dfrac{y-2x}{(x^2-xy+y^2)^2} \\ f_y = \dfrac{\partial f}{\partial y} = \dfrac{x-2y}{(x^2-xy+y^2)^2} \end{cases}$$

$$\nabla f(1,1) = (f_x, f_y) = (-1,-1)$$

West Direction

Thus, the slope along wester direction is: $(-1,-1)\cdot(-1,0) = 1$

$\vec{u} = (-1,0)$

The maximal slope is: $|(-1,-1)| = \sqrt{2}$

$$f(x,y) = \frac{1}{x^2-xy+y^2}.$$

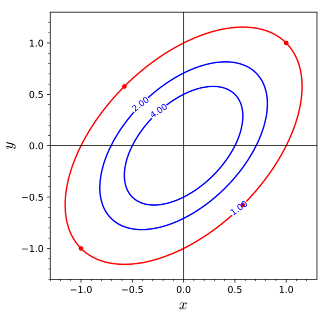
f) Express $f(tx,ty)$, $t \in \mathbb{R}$, in terms of $f(x,y)$. Use the result to describe the relation between the contours of f, and sketch the k-contour of f for $k=2$ and $k=4$; cf. d).

$$f(tx,ty) = \frac{1}{(tx)^2 - tx\cdot ty + (ty)^2} = \frac{1}{t^2} f(x,y)$$

Thus, for k-contour, $f(x,y) = k$, $\dfrac{f(x,y)}{k} = f(\sqrt{k}x, \sqrt{k}y)$

$\Rightarrow$ C_k is obtained from C_1 by scaling it with the factor $\dfrac{1}{\sqrt{k}}$

For $k=2,4$ the factors are $1/\sqrt{2}\approx 0.7$ and $1/2$, respectively; cf. picture.



Question 2 (ca. 6 marks)

Consider the differentiable map $G\colon D \to \mathbb{R}^2$, $D = \mathbb{R}^2 \setminus \{(0,0)\}$, defined by

$$G(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right).$$

a) Compute the Jacobi matrix $\mathbf{J}_G(x,y)$ and show that G is conformal.

☆Recall

Theorem

Let $f\colon D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, be a function with coordinate functions $f_1, \dots, f_m$ and $\mathbf{x} \in D^\circ$.

1. *If f is differentiable at $\mathbf{x}$ then f is continuous at $\mathbf{x}$.*
2. *If f is differentiable at $\mathbf{x}$ then the partial derivatives $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$ exist for $1 \le i \le m$, $1 \le j \le n$, and*

Jacobi Matrix $\longrightarrow$

$$\mathbf{J}_f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

- Conformal : *Note:* Recall that F is conformal in (x,y) iff $\mathbf{J}_F(x,y)^\mathsf{T}\mathbf{J}_F(x,y) = \lambda \mathbf{I}_2$ for some nonzero $\lambda \in \mathbb{R}$.

A function is conformal if it preserve angle locally.

Here in the question

$$G(x,y) = \left(\underbrace{\frac{x}{x^2+y^2}}_{f_1}, \underbrace{-\frac{y}{x^2+y^2}}_{f_2} \right).$$

$$J_G(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-x\cdot 2y}{(x^2+y^2)^2} \\ \frac{y\cdot 2x}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$J_G(x,y)^T J_G(x,y) = \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{y\cdot 2x}{(x^2+y^2)^2} \\ \frac{-x\cdot 2y}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{pmatrix} \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-x\cdot 2y}{(x^2+y^2)^2} \\ \frac{y\cdot 2x}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{(x^2+y^2)^2} & 0 \\ 0 & \frac{1}{(x^2+y^2)^2} \end{pmatrix} = \frac{1}{(x^2+y^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

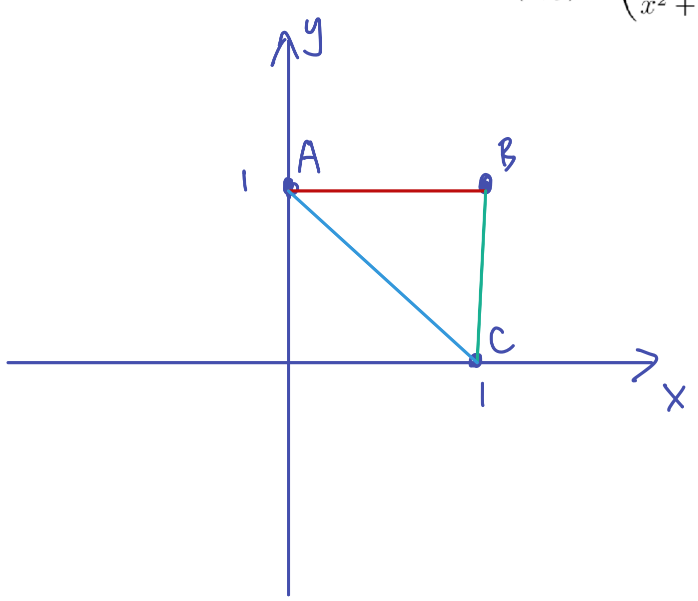
$= \lambda \mathbf{I}_2$ everywhere.

which means conformal.

b) Determine the G-image $S = G(\Delta)$ of the (solid) triangle Δ with vertices $(1,0)$, $(0,1)$, $(1,1)$, and graph Δ and the region S on paper (unit length at least 2 cm).

Hint: The G-images of the edges of Δ are circular arcs. Corresponding equations can be obtained by writing $G(x,y) = (u,v)$ and expressing $u^2 + v^2$ in terms of u, v. In the figure you should indicate the correspondence between edges and their G-images by using the same color (or line style such as "dashed", "dotted", etc.).

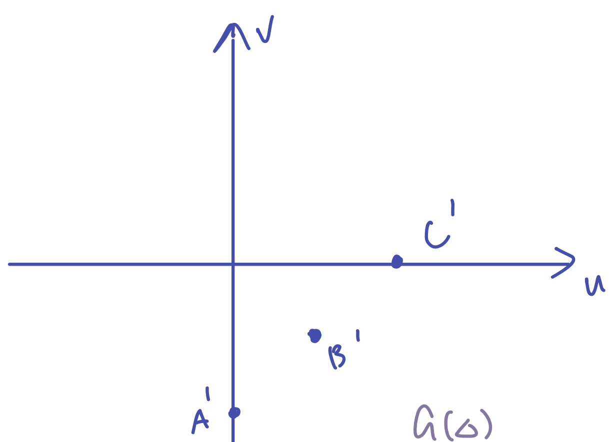
$$G(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right).$$



(Δ)

For three vertexs,

Δ		$G(\Delta)$
$(0,1)$	$\rightarrow$	$(0,-1)$
$(1,0)$	$\rightarrow$	$(1,0)$
$(1,1)$	$\rightarrow$	$(\frac{1}{2}, -\frac{1}{2})$



$G(\Delta)$

For segment BC,

$$G(1,y) = \left(\frac{1}{y^2+1}, -\frac{y}{y^2+1}\right)$$

$$u^2+v^2 = \left(\frac{1}{y^2+1}\right)^2 + \left(\frac{y}{y^2+1}\right)^2 = u$$

$$\Downarrow$$

$$\left(u-\tfrac{1}{2}\right)^2 + v^2 = \frac{1}{4}$$

For segment AC,

$$G(x, -x+1) = \left(\frac{x}{x^2+(1-x)^2}, \frac{x-1}{x^2+(1-x)^2}\right)$$

$$u^2+v^2 = \frac{1}{x^2+(1-x)^2} = u-v$$

$$\Downarrow$$

$$\left(u-\tfrac{1}{2}\right)^2 + \left(v+\tfrac{1}{2}\right)^2 = \frac{1}{2}$$

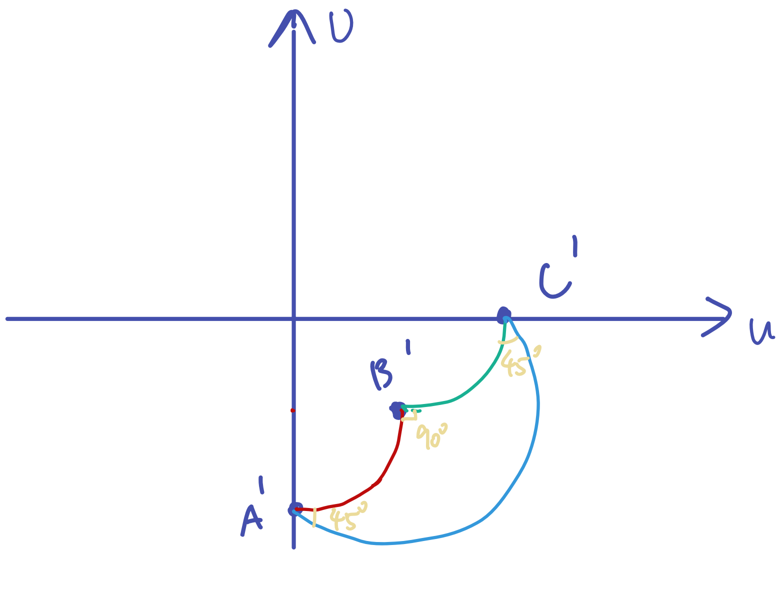
For segment AB, $y=1$

$$G(x,1) = \left(\frac{x}{x^2+1}, -\frac{1}{x^2+1}\right)$$

$$u^2+v^2 = \left(\frac{x}{x^2+1}\right)^2 + \left(\frac{1}{x^2+1}\right)^2 = \frac{1}{x^2+1} = -v$$

$$\Downarrow$$

$$u^2+\left(v+\tfrac{1}{2}\right)^2 = \frac{1}{4}$$



$G(\Delta)$

c) Is the figure obtained in b) compatible with the result in a)? Justify your answer!

Conformal: refers to a map or function that preserves angles.

Ans:

c) Since G is conformal in P_1, P_2, P_3, the angles between the sides of Δ (45°, 90°, 45°, respectively) must be the same as the angles between their image curves, which form the boundary of $G(\Delta)$. This is also visible in the figure of $G(\Delta)$. $\boxed{1}$

Question 3 (ca. 5 marks)

Consider the curve C in $\mathbb{R}^3$ parametrized by

$$\gamma(t) = \left(t^3 - t, 2\,t^3 + 1, 3\,t - 1\right), \quad t \in \mathbb{R}.$$

a) Is C contained in a plane? Justify your answer!

a) $x = t^3 - t$

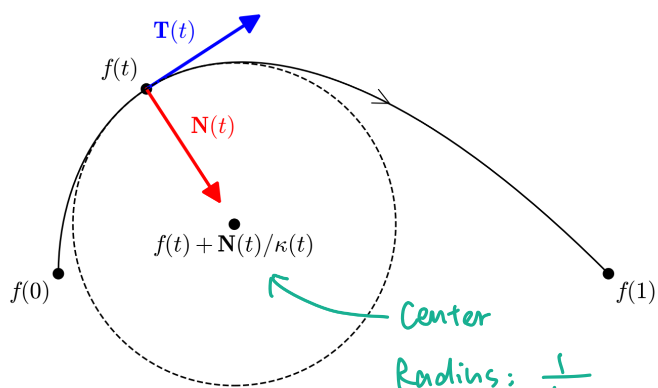
$y = 2t^3 + 1 \quad \Rightarrow \quad t^3 = \frac{y-1}{2}$

$z = 3t - 1 \quad \Rightarrow \quad t = \frac{z+1}{3}$

Thus, $x = \frac{y-1}{2} - \frac{z+1}{3} \quad \Longleftrightarrow \quad 6x - 3y + 2z + 5 = 0$ it's a plane!

b) Determine the center and radius of the osculating circle of C in $(0, 3, 2)$.

☆ Recall that for osculating circle:



Center

Radius: $\frac{1}{k(t)}$

☆ $k(t) = \dfrac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}$

$$\star \quad \underbrace{\vec{n}(t)} = \vec{\gamma}''(t) - \frac{\vec{\gamma}''(t) \cdot \vec{\gamma}'(t)}{|\vec{\gamma}'(t)|} \cdot \frac{\vec{\gamma}'(t)}{|\vec{\gamma}'(t)|}$$

note that it's not unit vector
but has the same direction as $\vec{N}(t)$

$$\gamma(t) = \left(t^3 - t, 2t^3 + 1, 3t - 1\right), \quad t \in \mathbb{R}.$$ At $(0,3,2)$, $t=1$

Here
$$\gamma'(t) = \begin{pmatrix} 3t^2 - 1 \\ 6t^2 \\ 3 \end{pmatrix}, \text{ and } \gamma'(1) = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

$$\gamma''(t) = \begin{pmatrix} 6t \\ 12t \\ 0 \end{pmatrix}, \text{ and } \gamma''(1) = \begin{pmatrix} 6 \\ 12 \\ 0 \end{pmatrix}$$

$$\text{Thus, } k(1) = \frac{|\gamma'(1) \times \gamma''(1)|}{|\gamma'(1)|^3} = \frac{42}{7^3} = \frac{6}{49}$$

Radius for osculating circle $r(1) = \frac{1}{k(1)} = \frac{49}{6}$

$$\vec{n}(1) = \vec{\gamma}''(1) - \frac{\vec{\gamma}''(1) \cdot \vec{\gamma}'(1)}{|\vec{\gamma}'(1)|} \cdot \frac{\vec{\gamma}'(1)}{|\vec{\gamma}'(1)|} = \frac{6}{7}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$$

Normal vector $\vec{N}(1)$ is a unit vector has the same direction of $\vec{n}(1)$

$$\vec{N}(1) = \frac{1}{7}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$$

Thus,

The center of osculating circle is

$$\vec{\gamma}(1) + \frac{\vec{N}(1)}{\kappa(1)} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \frac{7}{6}\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ \frac{16}{3} \\ -5 \end{pmatrix}$$

W23 *From a previous midterm*

The height and mass ("weight") of a basketball player are known to be $h_0 = 2.00\,\mathrm{m}$ and $m_0 = 100\,\mathrm{kg}$, respectively, with a possible error of 0.5 cm, respectively, 0.5 kg.

a) Find the linear approximation of the *body mass index* $B(m,h) = m/h^2$ near (m_0, h_0).

b) Using the Mean Value Theorem, state a (tight) rigorous upper bound for the (absolute) error when the body mass index of the basketball player is calculated from the known data.

Linear approximation: 4

$$\boxed{f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

a) Here $B(m,h) \approx B(m_0, h_0) + \frac{\partial B}{\partial m} \cdot \Delta m + \frac{\partial B}{\partial h} \cdot \Delta h$

$$= 25 + \frac{1}{4}\Delta m - 25\Delta h \quad [\mathrm{kg}\cdot\mathrm{m}^{-2}]$$

b)

lecture 21-23 handout

Error propagation

Usually we do not know the exact value of Δx_j but only a bound $|\Delta x_j| \le \delta_j$. In this case the Mean Value Theorem yields the upper bound

$$|\Delta y| \le \sum_{j=1}^{n} M_j \delta_j \quad \text{with} \quad M_j = \max_{\mathbf{x}' \in U} \left| \frac{\partial f}{\partial x_j}(\mathbf{x}') \right|,$$

where $U = \{\mathbf{x}' \in \mathbb{R}^n; \left|x_j' - x_j\right| \le \delta_j \text{ for } 1 \le j \le n\}$ is the "uncertainty region" of the input.

$\Delta m = 0.5\ kg \quad \Delta h = 0.005\ m$

Here $|\Delta B| = \left|\frac{1}{h'^2} \cdot \Delta m\right| + \left|\frac{-2m'}{h'^3} \Delta h\right|$ for some $m' \in [99.5, 100.5]$

$h' \in [1.995, 2.005]$

$$\therefore |\Delta B| \leq \frac{0.5}{1.995^2} + \frac{2 \times 100.5 \times 0.005}{1.995^3} \approx 0.25\ kg \cdot m^{-2}$$