- Partial derivatives 偏导

在一个热热的天，一名气象学家提出了 Heat index I 受 Temperature 和 humidity 影响：

Table 1 Heat index I as a function of temperature and humidity

Relative humidity (%)

Actual temperature (°C) T \ H	40	45	50	55	60	65	70	75	80
26	28	28	29	31	31	32	33	34	35
28	31	32	33	34	35	36	37	38	39
30	34	35	36	37	38	40	41	42	43
32	37	38	39	41	42	43	45	46	47
34	41	42	43	45	47	48	49	51	52
36	43	45	47	48	50	51	53	54	56

$I = f(T, H)$

For example, if we want to know how much I will change related to humidity when T is fixed to 30°C, the function $f(T, H)$ will become $f(30, H)$, which only has one independent variable

$\boxed{4}$ **Definitio** If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$: partial derivatives of f with respect to x

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$: partial derivatives of f with respect to y

partial derivative
的表示形式：

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

EXAMPLE 1 If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

$\parallel$ 16 $\qquad \parallel$ 8

$f_x(x,y) = x^3 + 2xy^3$

$f_y(x,y) = 3x^2y^2 - 4y$

- Second Partial Derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

For previous function $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$f_x(x,y) = x^3 + 2xy^3$$

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_{xx} = 3x^2 + 2y^3 \qquad f_{xy} = 6xy^2$$

equal

$$f_{yx} = 6xy^2 \qquad f_{yy} = 6x^2y - 4$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Theorem

Let $f\colon D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, be a function with coordinate functions $f_1, \ldots, f_m$ and $\mathbf{x} \in D^\circ$.

1. *If f is differentiable at $\mathbf{x}$ then f is continuous at $\mathbf{x}$.*
2. *If f is differentiable at $\mathbf{x}$ then the partial derivatives $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$ exist for $1 \le i \le m$, $1 \le j \le n$, and*

第1个函数分别对所有自变量求偏导

- Jacobi Matrix $\longrightarrow$

$$\mathbf{J}_f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

第m个函数分别对所有自变量求偏导

For example, let $f: D \to \mathbb{R}^2$, $D \subseteq \mathbb{R}^2$

$$\vec{F}(x,y) = \begin{pmatrix} x^2+y^2 \\ 2xy \end{pmatrix} \begin{matrix} f_1 \\ f_2 \end{matrix}$$

Then $$J_F(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}$$

- Tangent Plane

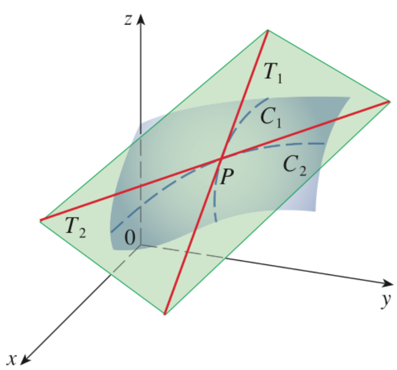


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

2 Equation of a Tangent Plane Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We know a plane can be expressed as:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$\Downarrow$ Dividing C

$$\frac{A}{C}(x-x_0) + \frac{B}{C}(y-y_0) + (z-z_0) = 0$$

the intersection with the plane $y=y_0$: $z - z_0 = -\frac{A}{C}(x-x_0)$

the intersection with the plane $x=x_0$: $z - z_0 = -\frac{B}{C}(y-y_0)$

$$\begin{cases} -\dfrac{A}{C} = \dfrac{z-z_0}{x-x_0} & \text{parcial derivative } f_x \\ -\dfrac{B}{C} = \dfrac{z-z_0}{y-y_0} \end{cases}$$

Thus, $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

- Differentiable

For a differentiable function $z = f(x,y)$,

Total differential is:

$$\boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

Differentiable Maps

What does $o(h)$ mean?

If $\lim_{h\to 0} R(h)/|h| = 0$

$R(h) = o(h)$

用来判断一个函数是否 differentiable:

Definition

Suppose $f\colon D \to \mathbb{R}^m$ is a map with domain $D \subseteq \mathbb{R}^n$ and $\mathbf{x}_0$ is an inner point of D. The map f is said to be *differentiable* at $\mathbf{x}_0$ if there exists a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + L(\mathbf{h}) + \mathrm{o}(\mathbf{h}) \quad \text{for} \quad \mathbf{h} \to \mathbf{0}. \qquad \text{(TD)}$$

If this is the case then the linear map L, which is uniquely determined, is called the *differential* of f at $\mathbf{x}_0$ and denoted by $\mathrm{d}f(\mathbf{x}_0)$.

same as

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-L(\mathbf{h})}{|\mathbf{h}|} = \mathbf{0} \in \mathbb{R}^m$$

Example ($f(x,y) = e^{xy}$)

$f_x = ye^{xy}$
$f_y = xe^{xy}$

This example has been included, because it is genuinly non-polynomial. Here we can argue as follows:

$$\begin{aligned}
e^{(x+h_1)(y+h_2)} - e^{xy} &= e^{xy+yh_1+xh_2+h_1h_2} - e^{xy} = e^{xy}\left(e^{yh_1+xh_2+h_1h_2} - 1\right) \\
&= e^{xy}\left(yh_1 + xh_2 + \text{terms of degree} \geq 2 \text{ in } \mathbf{h}\right) \\
&= \underbrace{(ye^{xy})}_{f_x} h_1 + \underbrace{(xe^{xy})}_{f_y} h_2 + o(\mathbf{h}).
\end{aligned}$$

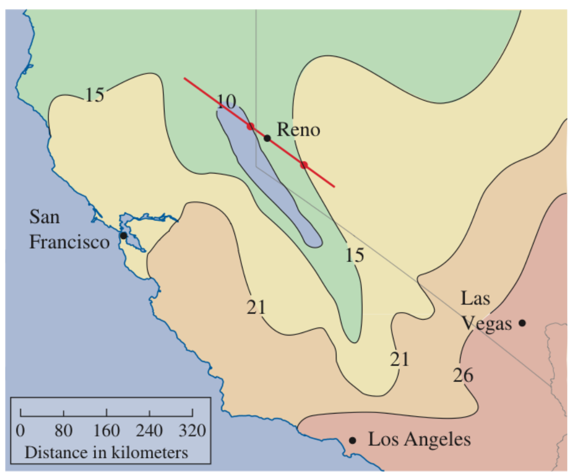
dz

$f_x dx$ $\quad$ $f_y dy$ $\quad$ it will vanish as $h \to 0$

- Directional Derivatives :

它的意义：

我们想知道在某个方向上函数值变化了多少



定义：

2 Definitio The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

- The gradient vector

8 Definitio If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Directional derivative VS. gradient :

$$\boxed{\underbrace{D_{\mathbf{u}} f(x, y)}_{\text{一个数值}} = \underbrace{\nabla f(x, y)}_{\text{一个向量}} \cdot \mathbf{u}}$$

以上讨论的都是2个自变量的，更多自变量的函数也同理.

limit $\underset{\longleftarrow}{\not\longrightarrow}$ continuous $\underset{\longleftarrow}{\not\longrightarrow}$ differentiable

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Worksheet 7

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Calculus III (Math 241)

W20 Show that the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g(0,0) = 0$ and

$$g(x,y) = \frac{xy^2}{x^2+y^2} \quad \text{for} \quad (x,y) \neq (0,0)$$

has directional derivatives at $(0,0)$ in every direction but is not differentiable at $(0,0)$.

$$\frac{\partial g}{\partial x} = g_x = \frac{y^2(x^2+y^2) - xy^2 \cdot 2x}{(x^2+y^2)^2}$$

$$g_x(x,y) = \lim_{h\to 0} \frac{g(x+h,y) - g(x,y)}{h}$$

$$\frac{\partial g}{\partial y} = g_y = \frac{2xy(x^2+y^2) - xy^2 \cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{\frac{(x+h)y^2}{x^2+y^2} - 0}{h} =$$

$$\frac{xy^2+hy^2}{(x^2+y^2)h}$$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$= \frac{xy^2}{(x^2+y^2)h} + \frac{y^2}{(x^2+y^2)}$$

W21 *From a previous midterm*

Consider the function $u\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by

$$u(x,y) = x + \frac{x}{x^2+y^2}.$$

a) Which symmetry properties does u have?

b) Describe the behaviour of $u(x,y)$ for $(x,y) \to (0,0)$, $x \geq 0$.
Hint: Use polar coordinates.

c) Show that u is differentiable and determine the differential $\mathrm{d}u$.

d) Show that there exists exactly one point (x_0, y_0) with $x_0 \geq 0$ at which $\mathrm{d}u$ vanishes.

e) Sketch the contour of u through (x_0, y_0).

a) $u(x,-y) = x + \frac{x}{x^2+y^2} = u(x,y) \Rightarrow u(x,y)$ 关于 x-z plane 对称

$u(-x,y) = -x - \frac{x}{x^2+y^2} = -u(x,y) \Rightarrow$ 关于 y 轴对称

b) let $x = r\cos\phi \quad y = r\sin\phi$

$u(x,y) = r\cos\phi + \frac{r\cos\phi}{r^2} = \cos\phi\left(r + \frac{1}{r}\right)$

As $(x,y) \to (0,0)$,

For $\cos\phi > 0$, $r \to 0$, $\frac{1}{r} \to \infty$, $u(x,y) \to \infty$

For $\cos\phi = 0$, $x = 0$, And $u(x,y)$ is continuous

$u(0,y) = \lim\limits_{y\to 0}\left(0 + \frac{0}{0+y^2}\right) = 0$

(c) $u_x = 1 + \dfrac{y^2 - x^2}{(x^2+y^2)^2}$ $\quad u_y = - \dfrac{2xy}{(x^2+y^2)^2}$

And $u(x,y)$ is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$,

Then $u(x,y)$ is differentiable

$$du = u_x\,dx + u_y\,dy$$
$$= \left(1 + \frac{y^2-x^2}{(x^2+y^2)^2}\right) dx - \frac{2xy}{(x^2+y^2)^2}\,dy$$

(d)

$$u_x = u_y = 0$$

if $x=0$, $u_x = 1 + \dfrac{y^2}{y^4} = 0$ no solution

if $y=0$, $u_x = 1 - \dfrac{x^2}{x^4} = 0$, $x = 1$

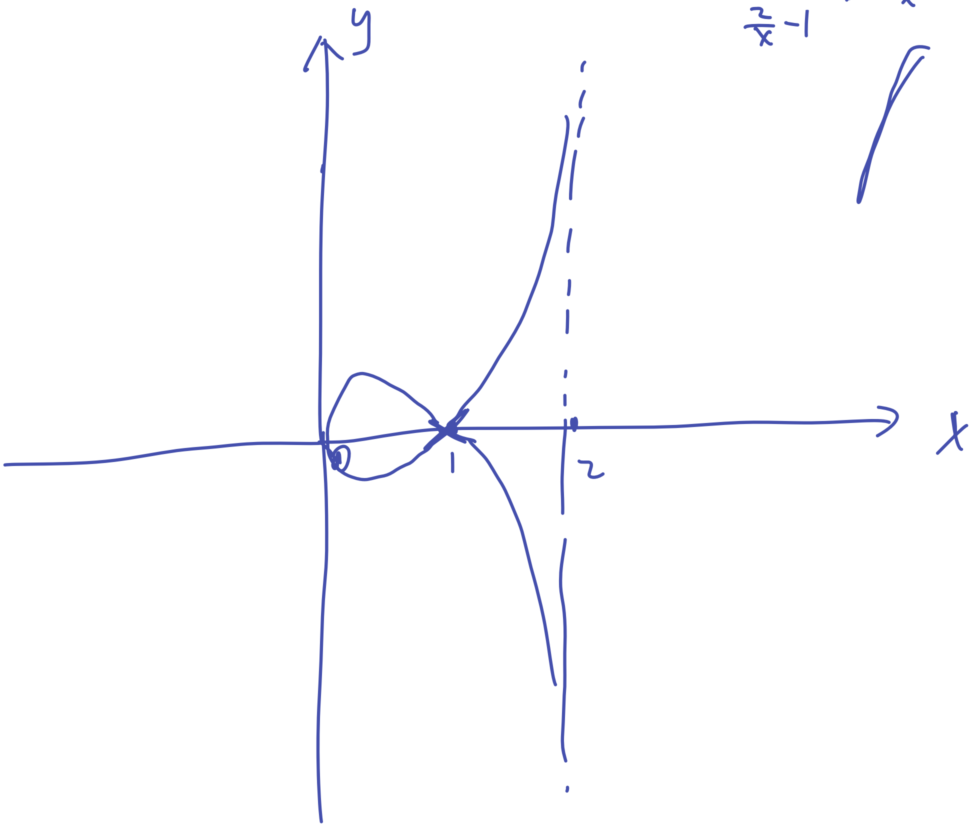
Thus, du vanishes at $(1,0)$

(e)

$$u(1,0) = 2$$

$$x + \frac{x}{x^2+y^2} = 2 \iff x^2 + y^2 = \frac{x}{2-x}$$

$$y^2 = \frac{x}{2-x} - x^2$$



$\dfrac{1}{\frac{2}{x}-1} - x$

$\dfrac{x-2x^2+x^3}{2-x}$

$y = \pm\sqrt[2]{\dfrac{x^3-2x^2+x}{2-x}}$ > 0 $\Rightarrow x<2$

$x(x^2-2x+1)$

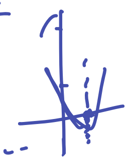
$y^2 = \dfrac{x(x-1)^2}{2-x} > 0$

$\Downarrow$

$\boxed{0<x<2}$

x^3-2x^2+x

$y = \sqrt{\dfrac{x(x-1)^2}{2-x}}$



$y' = \dfrac{\dfrac{(2-x)(3x^2-4x+1) \cdots}{(2-x)^2}}{2\sqrt{\dfrac{x(x-1)^2}{2-x}}} > 0$ when $x>1$

$\dfrac{6}{-6}$