迈向多变量函數

Some basic definitions first :

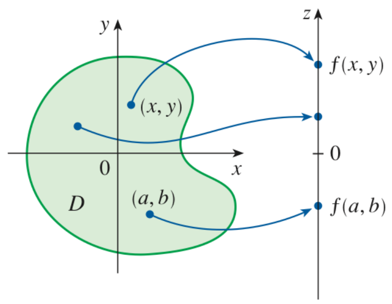


FIGURE 1

Definitio A **function *f* of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

x and y : independent variable

z : dependent variable

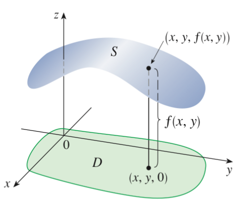


FIGURE 5

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definitio If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

Domain $\subseteq R^2$, the graph $\subseteq R^{2+1}$

Graphs

Definition

Let $f\colon D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, be a function. The *graph* of f is the point set

$$\mathrm{G}_f = \{(\mathbf{x}, f(\mathbf{x})); \mathbf{x} \in D\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$

The level sets = contour 等高線

Definitio $\boxed{\text{The \textbf{level curves}}}$ of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

• Limit

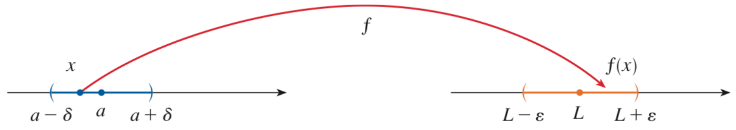
From Math 221

2 Precise Definition of a Limi Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$



For any given interval around L

We can find such interval around a, (可以不包括 a)

Such that f maps all points in $(a-\delta, a+\delta)$ into the interval $(L-\varepsilon, L+\varepsilon)$

Now with two independent variable:

1 Definitio Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

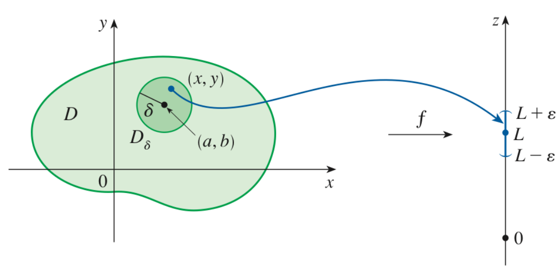


FIGURE 1

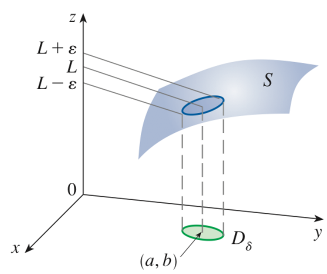


FIGURE 2

So, for a given function, how to determine if a limit exists or not?

① Use the definition directly. to show a limit exists.

Continuous $\Longrightarrow$ limit exisits

② Choose different paths to approach the given point to prove a limit does not exists.

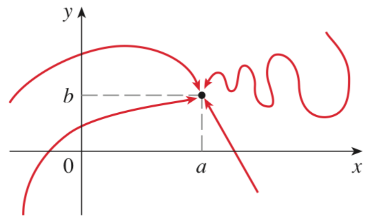


FIGURE 3
Different paths approaching (a, b)

$$\boxed{\text{If } f(x, y) \to L_1 \text{ as } (x, y) \to (a, b) \text{ along a path } C_1 \text{ and } f(x, y) \to L_2 \text{ as } (x, y) \to (a, b) \text{ along a path } C_2 \text{, where } L_1 \neq L_2 \text{, then } \lim_{(x, y) \to (a, b)} f(x, y) \text{ does not exist.}}$$

在计算 limit 可用的一些性质：

Properties of Limits

Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits. The Limit Laws listed in Section 2.3 can be extended to functions of two variables. Assuming that the indicated limits exist, we can state these laws verbally as follows:

Sum Law 1. The limit of a sum is the sum of the limits.

Difference Law 2. The limit of a difference is the difference of the limits.

Constant Multiple Law 3. The limit of a constant times a function is the constant times the limit of the function.

Product Law 4. The limit of a product is the product of the limits.

Quotient Law 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

So $\Downarrow$

A **polynomial function** of two variables (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of two polynomials. For instance,

$$p(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$q(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.