Math 241 Calculus III (Prof. Honold) **Midterm Exam 1** 2023/03/13

Name: ____________________ Student No.: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The area of the parallelogram spanned by $(1,a,2)$ and $(a,1,2)$ is equal to 9 for

   ☐ $a=-2$ ☐ $a=-1$ ☐ $a=0$ ☐ $a=1$ ☐ $a=2$

2. The distance from the point $(1,2,6)$ to the plane spanned by $(2,1,0)$, $(0,2,1)$, $(1,0,2)$ is equal to

   ☐ $\sqrt{2}$ ☐ $\sqrt{3}$ ☐ $2\sqrt{3}$ ☐ 6 ☐ $3\sqrt{2}$

3. The distance from the point $(2,2,6)$ to the line $2x+y=2y+z=1$ is equal to

   ☐ $\frac{1}{3}\sqrt{6}$ ☐ $\sqrt{6}$ ☐ $\frac{5}{3}\sqrt{6}$ ☐ $\frac{7}{3}\sqrt{6}$ ☐ $3\sqrt{6}$

4. $\mathbf{A}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}$ satisfy $\mathbf{Av}=-\mathbf{v}$ if

   ☐ $\phi=22.5^\circ$ ☐ $\phi=45^\circ$ ☐ $\phi=67.5^\circ$ ☐ $\phi=90^\circ$ ☐ $\phi=112.5^\circ$

5. The smallest distance d^* from the curve $f(t)=(1,0,0)+t(1,-1,0)+t^2(0,1,-1)$, $t\in\mathbb{R}$ to the origin satisfies

   ☐ $d^*=0$ ☐ $d^*\in\left(0,\frac{1}{2}\right)$ ☐ $d^*=\frac{1}{2}$ ☐ $d^*\in\left(\frac{1}{2},1\right)$ ☐ $d^*=1$

6. The maximum curvature of the curve $f(t)$ in Question 5 is

   ☐ $\frac{4}{3}\sqrt{2}$ ☐ $16\sqrt{3}$ ☐ $2\sqrt{6}$ ☐ $\frac{3}{4}\sqrt{3}$ ☐ $8\sqrt{3}$

7. The tangent to the curve $g(t)=(t,t^2,t^4)$, $t\in\mathbb{R}$ in the point $(1,1,1)$ intersects the plane $ax+y-2z=2023$ unless

   ☐ $a=10$ ☐ $a=1$ ☐ $a=6$ ☐ $a=3$ ☐ $a=0$

8. For the twisted cubic $f(t)=(t,t^2,t^3)$, $t\in\mathbb{R}$ the unit normal vector $\mathbf{N}(1)$ is a positive multiple of

   ☐ $(-11,8,9)$ ☐ $(11,8,-9)$ ☐ $(11,-8,9)$ ☐ $(-11,8,-9)$
   ☐ $(-11,-8,9)$

9. The arc length of the curve $g(t)=\big(3t\sin(2t),4t^{3/2},3t\cos(2t)\big)$, $t\in[0,5]$ is

   ☐ 30 ☐ 45 ☐ 60 ☐ 75 ☐ 90

10. For a differentiable curve $\gamma=\gamma(t)$ in $\mathbb{R}^3$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left(|\gamma|^2\gamma\right)$ is equal to

   ☐ $2|\gamma|\,|\gamma'|\,\gamma+|\gamma|^2\gamma'$ ☐ $2\gamma+|\gamma|^2\gamma'$ ☐ $2(\gamma\cdot\gamma')\gamma+(\gamma\cdot\gamma)\gamma'$
   ☐ $2|\gamma|\gamma+|\gamma|^2\gamma'$ ☐ $2|\gamma|\gamma'+(\gamma\cdot\gamma)\gamma'$

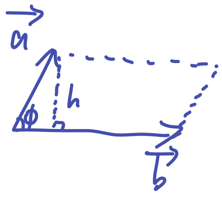
Time allowed: 45 min CLOSED BOOK ***Good luck!***

Note prepared by Jiarui Yu.

1. The area of the parallelogram spanned by $(1,a,2)$ and $(a,1,2)$ is equal to 9 for

☑ $a=-2$ ☐ $a=-1$ ☐ $a=0$ ☐ $a=1$ ☐ $a=2$

let $\vec{a} = (1, a, 2)$ $\vec{b} = (a, 1, 2)$



Area $= |\vec{b}| \cdot h = |\vec{a}||\vec{b}| \cdot \sin\phi$
$= \vec{a} \times \vec{b}$

Area $= |\vec{a} \times \vec{b}|$

$$= \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & a & 2 \\ a & 1 & 2 \end{matrix} \right|$$

$$= |\langle 2a-2, 2a-2, 1-a^2 \rangle| = 9$$

$$\therefore (2a-2)^2 + (2a-2)^2 + (1-a^2)^2 = 81$$

$$a = -2$$

2. The distance from the point $(1,2,6)$ to the plane spanned by $(2,1,0)$, $(0,2,1)$, $(1,0,2)$ is equal to

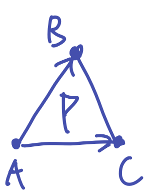
☐ $\sqrt{2}$ ☐ $\sqrt{3}$ ☑ $2\sqrt{3}$ ☐ 6 ☐ $3\sqrt{2}$

☆ Recall How to calculate representations of a plane.

> **7** A **scalar equation of the plane** through point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

What we need : $\begin{cases} \text{normal vector } \vec{n} \text{ of the plane} \\ \text{A point on the plane} \end{cases}$ $\Rightarrow$ the representation of a plane

For this question, let $\vec{A} = (2, 1, 0)$ $\vec{B} = (0, 2, 1)$ $\vec{C} = (1, 0, 2)$



$\vec{AB} = (-2, 1, 1)$

$\vec{AC} = (-1, -1, 2)$

Then $\vec{n} = \vec{AB} \times \vec{AC} = \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ -1 & -1 & 2 \end{matrix} \right| = (3, 3, 3) = 3(1, 1, 1)$

Also, we know that the plane passes point $C(1,0,2)$

The representation of the plane is: $P: 1(x-1)+1(y-0)+1(z-2)=0$

$$\Downarrow$$

$$\boxed{x+y+z-3=0}$$

☆ Recall the distance from a point to a plane.

9 The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

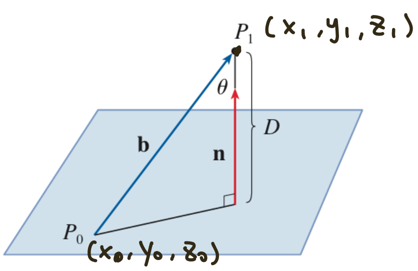
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

In this question, P_1 is $P_1(1,2,6)$

Thus, $D = \dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|1+2+6-3|}{\sqrt{1+1+1}} = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$

Alternatively, we can also use the following formula directly:

$\vec{n} = (a,b,c)$ Normal vector of the plane



$$D = |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} .$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

3. The distance from the point $(2,2,6)$ to the line $2x+y=2y+z=1$ is equal to

☐ $\frac{1}{3}\sqrt{6}$ ☐ $\sqrt{6}$ ☑ $\frac{5}{3}\sqrt{6}$ ☐ $\frac{7}{3}\sqrt{6}$ ☐ $3\sqrt{6}$

Step 1. get the representation of the line :

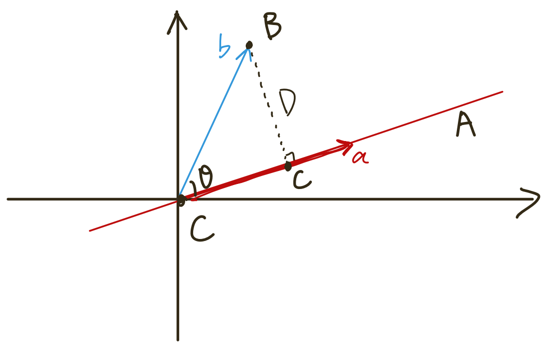
Set $x = c$

$$\because \begin{cases} 2x+y=1 \\ 2y+z=1 \end{cases} \Rightarrow \begin{cases} y = 1-2c \\ z = -1+4c \end{cases}$$

$$\text{Thus, } L = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \\ 1-2c \\ -1+4c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \quad \text{where } c \in \mathbb{R}$$

Sep2. Calculate the distance.

☆ Recall: ⑤ pointB to lineA : $D = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$ where $\vec{a}$ is the direction vector of the line, $\vec{b}$ is the vector start from a point on line A, end at point B



In this question, $\vec{a} = (1, -2, 4)$

$$\vec{b} \text{ can be: } \underbrace{(2,2,6)}_{\text{the point we are interested in}} - \underbrace{(0,1,-1)}_{\text{An arbitrary point on the line.}} = (2,1,7)$$

$$\text{Thus, } D = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 2 & 1 & 7 \end{vmatrix}}{\sqrt{1+4+16}} = \frac{|(-18, 1, 5)|}{\sqrt{21}} = \frac{\sqrt{350}}{\sqrt{21}} = \frac{5}{3}\sqrt{6}$$

4. $\mathbf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$ satisfy $\mathbf{A}\mathbf{v} = -\mathbf{v}$ if

☐ $\phi = 22.5°$ ☐ $\phi = 45°$ ☐ $\phi = 67.5°$ ☐ $\phi = 90°$ ☑ $\phi = 112.5°$

☆ Recall: switching between two different coordinate systems.

Observation

$O' \triangleq \mathbf{p} = (p_1, p_2)$ for some $\mathbf{p} \in \mathbb{R}^2$ and the new unit coordinate directions are represented by $\mathbf{p} + \mathbb{R}\mathbf{a}$, $\mathbf{p} + \mathbb{R}\mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ orthogonal and of the same length.

$\Longrightarrow$ A point Q with new coordinates (x_1', x_2') has old coordinates

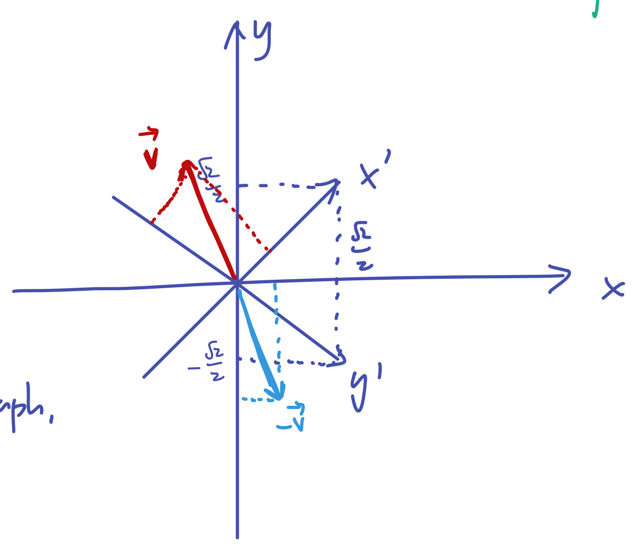
☆ $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{p} + x_1'\mathbf{a} + x_2'\mathbf{b} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + x_1'\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + x_2'\begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$

From handout 1-3

In this question, let $\boxed{-\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}$ old coordinate

$$A\vec{v} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\underbrace{\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}}_{\text{new coordinate}} = \begin{pmatrix} a_{11}v_1' + a_{12}v_2' \\ a_{21}v_1' + a_{22}v_2' \end{pmatrix} = v_1'\underbrace{\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}_{\text{new x-axis}} + v_2'\underbrace{\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}_{\text{new y-axis}}$$

We have $A = \Big(\underbrace{\begin{matrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{matrix}}_{\text{new x-axis}} \quad \underbrace{\begin{matrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{matrix}}_{\text{new y-axis}}\Big)$



For $\phi = 112.5°$, As shown in the graph,

$Av = -v$

Alternatively, Recall Reflection matrix $S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = A$

Here $\phi = 45°$, the reflection axis should be $y = \tan(22.5°)x$

Thus, for $\vec{v} = (\cos\theta, \sin\theta)$, $\theta = 22.5° + 90° = 112.5°$

5. The smallest distance d^* from the curve $f(t) = (1,0,0) + t(1,-1,0) + t^2(0,1,-1),\ t \in \mathbb{R}$ to the origin satisfies

☐ $d^* = 0$ ☐ $d^* \in (0, \frac{1}{2})$ ☐ $d^* = \frac{1}{2}$ ☑ $d^* \in (\frac{1}{2}, 1)$ ☐ $d^* = 1$

$f(t)$ lies on the plane $P = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $\quad c_1, c_2 \in R$

$$\Downarrow$$

$$P: x + y + z - 1 = 0$$

Distance from the origin to the plane

Thus $D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{1}{\sqrt{3}} > \dfrac{1}{2}$

Then, we need to determine if there's a point on the curve makes $d^* < 1$

$$f(t) = \begin{pmatrix} 1+t \\ -t+t^2 \\ -t^2 \end{pmatrix}$$

$$d = \sqrt{(1+t)^2 + (-t+t^2)^2 + (-t^2)^2} = \sqrt{2t^4 - 2t^3 + 2t^2 + 2t + 1}$$

let $t = -\frac{1}{2}$, $\quad d = \frac{7}{8} < 1$

Thus, $d^* \in (\frac{1}{2}, 1)$

6. The maximum curvature of the curve $f(t)$ in Question 5 is

☑ $\frac{4}{3}\sqrt{2}$ ☐ $16\sqrt{3}$ ☐ $2\sqrt{6}$ ☐ $\frac{3}{4}\sqrt{3}$ ☐ $8\sqrt{3}$

Recall:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Here $\vec{r}(t) = \langle t+1,\ t^2 - t,\ -t^2 \rangle$

$\vec{r}'(t) = \langle 1,\ 2t-1,\ -2t \rangle$

$\vec{r}''(t) = \langle 0,\ 2,\ -2 \rangle$

Thus, $\kappa = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \dfrac{|\langle 2, 2, 2 \rangle|}{(8t^2 - 4t + 2)^{\frac{3}{2}}} = \dfrac{2\sqrt{3}}{(8t^2 - 4t + 2)^{\frac{3}{2}}}$

let $g(t) = 8t^2 - 4t + 2$

let $g'(t) = 16t - 4 = 0 \Rightarrow t^* = \frac{1}{4}$, κ_{max}

$\kappa_{max} = \kappa(\frac{1}{4}) = \frac{4}{3}\sqrt{2}$

7. The tangent to the curve $g(t) = (t, t^2, t^4)$, $t \in \mathbb{R}$ in the point $(1,1,1)$ intersects the plane $ax + y - 2z = 2023$ unless

☐ $a = 10$ ☐ $a = 1$ ☑ $a = 6$ ☐ $a = 3$ ☐ $a = 0$

☆ The tangent line to a curve at $g(t)$: $T = g(t) + \mathbb{R} g'(t)$

$g(t) = (t, t^2, t^4)$ $g(1) = (1,1,1)$

$g'(t) = (1, 2t, 4t^3)$ $g'(1) = (1, 2, 4)$

Thus, the tangent line at $(1,1,1)$ is $T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

T has direction vector: $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

For the plane $ax + y - 2z = 2023$, normal vector $\vec{n} = \begin{pmatrix} a \\ 1 \\ -2 \end{pmatrix}$

Iff $\vec{a} \perp \vec{n}$, they have no intersection.

Thus, $\vec{a} \cdot \vec{n} = a + 2 - 8 = 0 \Rightarrow a = 6$

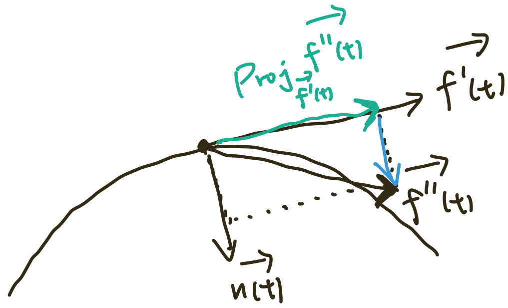
8. For the twisted cubic $f(t) = (t, t^2, t^3)$, $t \in \mathbb{R}$ the unit normal vector $\mathbf{N}(1)$ is a positive multiple of

☐ $(-11, 8, 9)$ ☐ $(11, 8, -9)$ ☐ $(11, -8, 9)$ ☐ $(-11, 8, -9)$ ☐ $(-11, -8, 9)$

☆ $\underbrace{\vec{n}(t)} = \vec{f''}(t) - \dfrac{\vec{f''}(t) \cdot \vec{f'}(t)}{|\vec{f'}(t)|} \cdot \dfrac{\vec{f'}(t)}{|\vec{f'}(t)|}$ ①

note that it's not unit vector but has the same direction as $\vec{N}(t)$

How to understand equation ① :



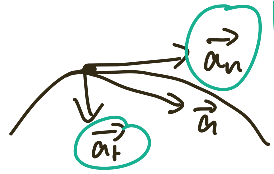
For a particle moves along $\vec{f}(t)$

$$\vec{v}(t) = \vec{f'}(t)$$

$$\vec{a}(t) = \vec{f''}(t)$$

For $\vec{a}(t)$, it can be decomposed to

$$\underbrace{\vec{a}_r}_{\text{centripetal acceleration}} + \underbrace{\vec{a}_n}_{\text{tangent acceleration.}}$$



has same direction of velocity. i.e. $\vec{f'}(t)$

has same direction of $\vec{n}(t)$

$$\vec{a}_r = \vec{a} - \vec{a}_n$$

can be seen as $\vec{f''}(t)$

vector projection of $\vec{f''}(t)$ onto $\vec{f'}(t)$

$$\boxed{\frac{\vec{f''}(t) \cdot \vec{f'}(t)}{|\vec{f'}(t)|} \cdot \frac{\vec{f'}(t)}{|\vec{f'}(t)|}}$$

Thus, in this question here,

$$\vec{n}(1) = \vec{f''}(1) - \frac{\vec{f''}(1) \cdot \vec{f'}(1)}{|\vec{f'}(1)|} \cdot \frac{\vec{f'}(1)}{|\vec{f'}(1)|}$$

9. The arc length of the curve $g(t) = \left(3t\sin(2t), 4t^{3/2}, 3t\cos(2t)\right)$, $t \in [0,5]$ is

☐ 30 ☐ 45 ☐ 60 ☐ 75 ☑ 90

A

$$\boxed{L = \int_a^b |\mathbf{r}'(t)|\,dt}$$

Here $g'(t) = \left(3\sin(2t) + 6t\cos(2t),\ 6t^{\frac{1}{2}},\ 3\cos(2t) - 6t\sin(2t)\right)$

$$L = \int_0^5 |\vec{g'(t)}|\,dt = \int_0^5 \sqrt{9 + 36t + 36t^2}\,dt = \int_0^5 3\sqrt{1+4t+4t^2}\,dt$$

$$= 3\int_0^5 (1+2t)\,dt = 3(t^2 + t)\Big]_0^5 = 90$$

10. For a differentiable curve $\gamma = \gamma(t)$ in $\mathbb{R}^3$ the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\left(|\gamma|^2\gamma\right)$ is equal to

☐ $2|\gamma|\,|\gamma'|\,\gamma + |\gamma|^2\gamma'$ ☐ $2\gamma + |\gamma|^2\gamma'$ ☑ $2(\gamma\cdot\gamma')\gamma + (\gamma\cdot\gamma)\gamma'$

☐ $2|\gamma|\gamma + |\gamma|^2\gamma'$ ☐ $2|\gamma|\gamma' + (\gamma\cdot\gamma)\gamma'$

$$\frac{d}{dt}(|\gamma|^2\gamma) = \frac{d}{dt}\left[(\gamma\cdot\gamma)\cdot\gamma\right] = (\gamma'\gamma + \gamma\gamma')\gamma + (\gamma\cdot\gamma)\cdot\gamma'$$

$$= 2(\gamma'\gamma)\gamma + (\gamma\cdot\gamma)\cdot\gamma'$$

$(\gamma \cdot \gamma)^{\frac{1}{2}}$

$$\frac{\gamma'|\gamma| - \gamma|\gamma|'}{|\gamma|^2} = \frac{\gamma'|\gamma|}{|\gamma|^2} - \frac{\gamma|\gamma|'}{|\gamma|^2}$$

$$|r| = (\sqrt{r \cdot r})' = \frac{r \cdot r' + r' \cdot r}{2\sqrt{r \cdot r}} = \frac{\gamma \cdot \gamma'}{|r|}$$

$$\boxed{|r|' = \frac{r' \cdot r}{|r|}}$$

$$\frac{d}{dt}\frac{r}{|r|} = \frac{r'|r| - |r|' \cdot r}{|r|^2} = \frac{r'|r| - \frac{r' \cdot r}{|r|} \cdot r}{|r|^2}$$