Differentiation of Curves

The definition of the derivative of a real-valued function generalizes almost verbatim to the case of parametric curves.

Definition

Let $f\colon I \to \mathbb{R}^n$ be a parametric curve.

1. We say that f is *differentiable* at $t_0 \in I$ if the limit

$$f'(t_0) = \lim_{h\to 0} \frac{1}{h}\big(f(t_0+h)-f(t_0)\big) = \lim_{h\to 0} \begin{pmatrix} \frac{1}{h}(f_1(t_0+h)-f_1(t_0)) \\ \vdots \\ \frac{1}{h}(f_n(t_0+h)-f_n(t_0)) \end{pmatrix}$$

   exists, and *differentiable (per se)* if this limit exists for all $t_0 \in I$.

$$f'(t_0) = \begin{pmatrix} f_1'(t_0) \\ \vdots \\ f_n'(t_0) \end{pmatrix}.$$

2. Let $D \subseteq I$ be the set of all t_0 at which f is differentiable according to (1). The parametric curve $D \to \mathbb{R}^n$, $t \mapsto f'(t)$ is called the *derivative* of f.

Examples

1. The twisted cubic $\boxed{f(t) = (t, t^2, t^3)}$, $t \in \mathbb{R}$, is differentiable with $f'(t) = (1, 2t, 3t^2)$.

   $1 \quad 2t \quad 3t^2$

Integration of Curves

Definition

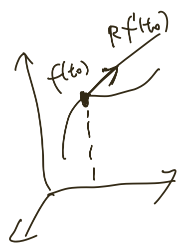
Curves are *integrated coordinate-wise*, i.e., if $f(t) = \big(f_1(t), f_2(t), \dots, f_n(t)\big)$ and the f_i are integrable over $[a, b]$ then

$$\int_a^b f(t)\mathrm{d}t = \left(\int_a^b f_1(t)\mathrm{d}t, \int_a^b f_2(t)\mathrm{d}t, \dots, \int_a^b f_n(t)\mathrm{d}t\right).$$

- Tangent Line :

Definition

If $f\colon I \to \mathbb{R}^n$ is differentiable at $t_0 \in I$ and $f'(t_0) \neq \mathbf{0}$, the line $f(t_0) + \mathbb{R}f'(t_0)$ (i.e., the line through $f(t_0)$ with direction vector $f'(t_0)$) is called the *tangent line* to f at t_0.



EXAMPLE 3 Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

At $(0, 1, \frac{\pi}{2})$. $\quad t = \frac{\pi}{2} \qquad f'(t) = \langle -2\sin t, \cos t, 1 \rangle$

$f'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

tangent line : $\quad L = f(\frac{\pi}{2}) + \mathbb{R} f'(\frac{\pi}{2}) = \begin{pmatrix} 0 \\ 1 \\ \frac{\pi}{2} \end{pmatrix} + \mathbb{R}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

• Arc Length

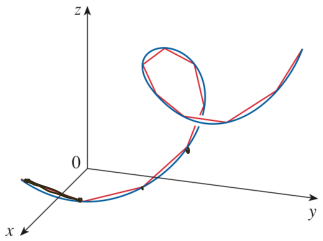


FIGURE 1
The length of a space curve is the limit of lengths of approximating polygonal paths.

Definition:

① $$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

② $$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

★ $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$

$a \le t \le b$

f', g', h' are continuous

EXAMPLE 1 Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

$L = 2\sqrt{2}\pi$

$0 - 2\pi$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= \sqrt{2}\, t \Big]_0^{2\pi} = 2\sqrt{2}\pi$$

- Matrix Inversion . To find A^{-1}

Method 1. $(A \mid I) \xrightarrow{\text{Gaussia elimination}} (I \mid B)$, Then $A^{-1} = B$

Method 2.

$$A^{-1} = \operatorname{adj}(A)/\det(A),$$

where

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

(For 3x3 Matrix)

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be the 3 x 3 matrix. The inverse matrix is:

$$A^{-1} = \frac{1}{|A|} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{vmatrix}$$

Example

We compute the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & -6 \\ -1 & -1 & 3 \\ 4 & 3 & 0 \end{bmatrix}$.

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{rrr|rrr} 2 & 1 & -6 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3=R3-2\,R1]{R2=R2+\frac{1}{2}R1} \left[\begin{array}{rrr|rrr} 2 & 1 & -6 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 12 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R2=-2\,R2]{R1=\frac{1}{2}R1} \left[\begin{array}{rrr|rrr} 1 & \frac{1}{2} & -3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 12 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R3=R3-R2]{R1=R1-\frac{1}{2}R2} \left[\begin{array}{rrr|rrr} 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 12 & -1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R3=\frac{1}{12}R3]{R1=R1+\frac{1}{4}R3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{3}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{array}\right]$$

Consequently, $\mathbf{A}^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{3}{2} & \frac{1}{4} \\ -1 & -2 & 0 \\ -\frac{1}{12} & \frac{1}{6} & \frac{1}{12} \end{bmatrix}$.

☆ For orthogonal Matrix Q, $Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I,$ which means $Q^{T} = Q^{-1}$

{ columns and rows are mutually orthogonal
all columns and rows are unit vectors.