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Today's topic:

- Distance
- Determinants
- Cross product and applications

Distance

1. point to point
2. point to plane
3. two parallel planes
4. line to line
5. point to line

① point to point :

The distance between points $p(p_1, p_2, p_3)$ and $q(q_1, q_2, q_3)$ is

$$d(\mathbf{p}, \mathbf{q}) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}.$$

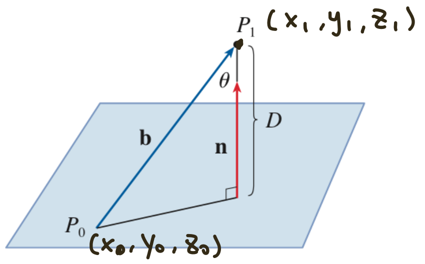
② point to plane :

**9** The distance $D$ from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

proof :

$\vec{n} = (a, b, c)$ Normal vector of the plane




$$D = |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$$

$$= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} .$$

$$= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} =$$

③ two parallel planes : We can choose any point on one plane, then calculate the distance between that point to another plane.

④ line to line : Two skew lines can be viewed as lying on two parallel plane. Then calculate the distance between two parallel planes.

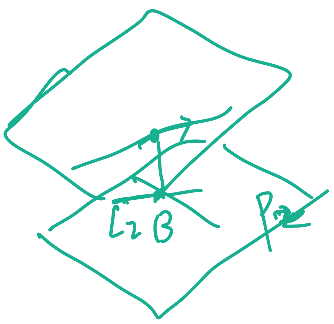
**EXAMPLE 9** In Example 3 we showed that the lines

$$L_1: \quad x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$

$$L_2: \quad x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

are skew. Find the distance between them.

We need to find the expression of a plane and a point on another plane.




$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$L_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \vec{a}$$

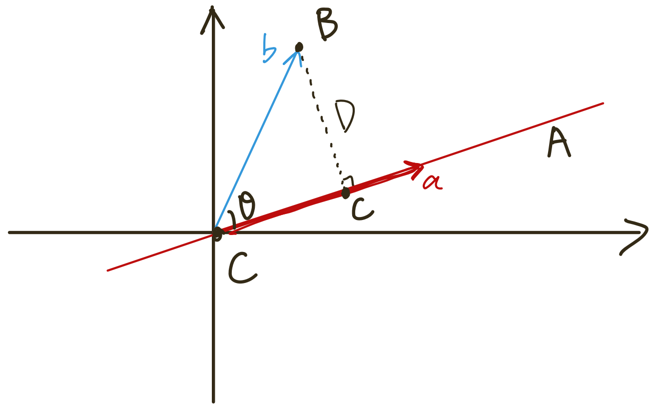
$$L_2 = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \vec{b}$$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = \langle 13, -6, -5 \rangle$$

let $s=0$ $\quad B(0,3,-3)$

$$P_2: 13(x-0)-6(y-3)-5(z+3)=0$$

⑤ point B to line A : $D = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$ where $\vec{a}$ is the direction vector of the line, $\vec{b}$ is the vector start from a point on line A, end at point B




Example : $\underbrace{\boxed{(4, 1, -2);}}_{B}$ $\underbrace{\boxed{x = 1 + t, \quad y = 3 - 2t, \quad z = 4 - 3t}}_{A}$

$$A = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + t \underbrace{\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}}_{\vec{a}}$$

let $t=0$ , $C(1, 3, 4)$

$$\vec{b} = B - C = (4, 1, -2) - (1, 3, 4) = (3, -2, -6)$$

$$D = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 3 & -2 & -6 \end{vmatrix}}{\sqrt{1+4+9}} = \frac{|\langle 6, -3, 4 \rangle|}{\sqrt{14}} = \frac{\sqrt{36+9+16}}{\sqrt{14}}$$

1 2 3

# Determinants.

$$A = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}$$

- General form:

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij}) \quad \text{(for Row } i\text{)}$$
$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij}), \quad \text{(for Column } j\text{)}$$

where $A_{ij}$ denotes the $(n-1)\times(n-1)$ submatrix of $\mathbf{A}$ obtained by deleting Row $i$ and Column $j$.

- For 2×2 and 3×3 matrix:

**Definition**

1. For $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$ we define its *determinant* as

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

2. For $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathbb{R}^{3\times 3}$ we define

$$\det(\mathbf{A}) = +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}.$$

Instead of $\det(\mathbf{A})$ one also writes $|\mathbf{A}|$.

# Cross product :

- Definition:

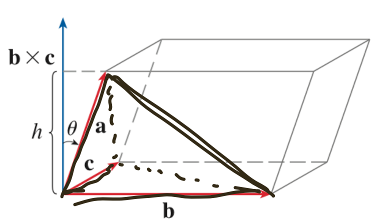
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1. \end{pmatrix}$$

- The property of $|\vec{a}\times\vec{b}|$ :

**9 Theorem** If $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a}\times\mathbf{b}$ is given by

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

- Application: using cross product to calculate the volume of tetrahedron 四面体.




**14** The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product:

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$|\vec{b}\times\vec{c}| = |\vec{b}||\vec{c}|\sin\theta \qquad |\vec{b}\times\vec{c}|\cdot\vec{a}$$

$$= |\vec{b}\times\vec{c}|\,|\vec{a}|\cdot\cos\phi$$

$h$

Example:

1. The volume of the pyramid ("tetrahedron") with vertices $\underset{A}{(0,1,1)}, \underset{B}{(1,0,1)}, \underset{C}{(1,1,0)}, \underset{D}{(1,2,3)}$ is equal to

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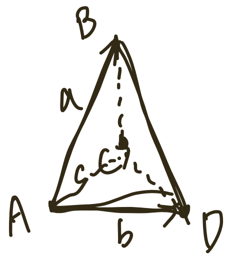


$$\vec{a} = B - A = (1,-1,0)$$
$$\vec{b} = D - A = (1,1,2)$$
$$\vec{c} = C - A = (1,0,-1)$$

$$V = \frac{1}{6}|\vec{a}\cdot(\vec{b}\times\vec{c})| = \frac{1}{6}|(1,-1,0)\cdot(-1,3,-1)|$$
$$= \frac{1}{6}\times|-4| = \frac{2}{3}$$