Cross Product
The angle between two vectors
Representations of Lines and Planes in R^3

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Cross Product

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

8 Theorem The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

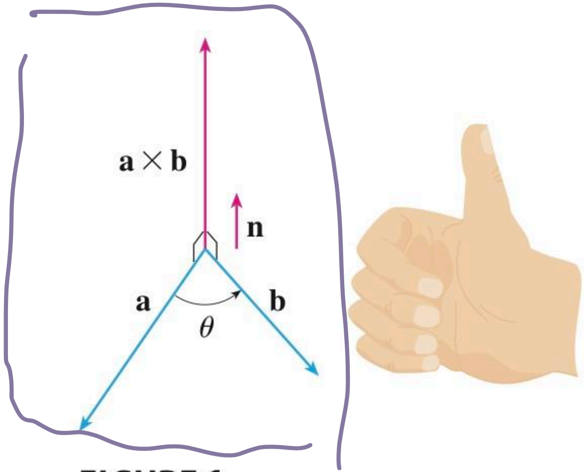


FIGURE 1
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

EXAMPLE 1 If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

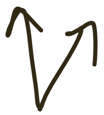
$$= (-15 - 28)\,\mathbf{i} - (-5 - 8)\,\mathbf{j} + (7 - 6)\,\mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k} \qquad \blacksquare$$

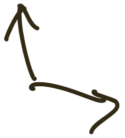
The angle between two vectors

- Generally, the *angle* $\phi \in [0, \pi]$ between vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ a point $\mathbf{p} \in l_1 \cap l_2$ is determined by

$$\cos \phi = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

The angle is *acute* if $\mathbf{a} \cdot \mathbf{b} > 0$ and *obtuse* if $\mathbf{a} \cdot \mathbf{b} < 0$.





Representations of Lines and Planes in R^3

There exists alternative representations for planes/lines:

- **Equational Representation**
- **Parametric Representation**

Equational Representation

Planes

Every plane in $\mathbb{R}^3$ is the solution set of a single linear equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

with $a_1, a_2, a_3, b \in \mathbb{R}$ and not all a_i equal to zero (i.e., $(a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$).

Conversely, every such point set is a plane in $\mathbb{R}^3$.

Linear equations $a_1 x_1 + a_2 x_2 + a_3 x_3 = b$ and $a'_1 x_1 + a'_2 x_2 + a'_3 x_3 = b'$ represent the same plane iff the vectors (a_1, a_2, a_3, b) and (a'_1, a'_2, a'_3, b') are scalar multiples of each other.

Equational Representation

Lines

Every line in $\mathbb{R}^3$ is the intersection of 2 planes (in many ways) and hence the solution set of a system of 2 linear equations.

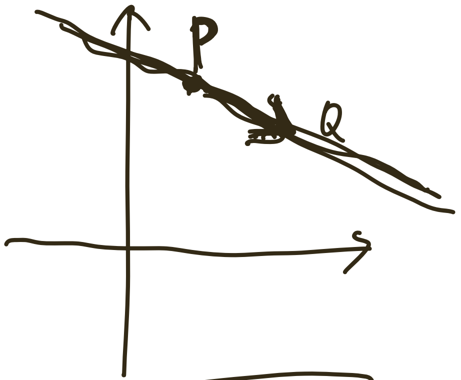
Conversely, the intersection of any 2 distinct planes in $\mathbb{R}^3$ is a line.

$$\boxed{\begin{aligned} a1x1 + a2x2 + a3x3 &= b1 \quad (1) \\ a4x1 + a5x2 + a6x3 &= b2 \quad (2) \end{aligned}}$$

with a1,a2,a3,a4,a5,a6,b1,b2 belongs to R and not all ai equals to zero.

Parametric Representation

Lines



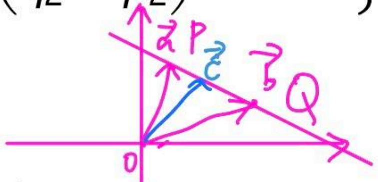
$$\boxed{l = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \mathbb{R} \cdot \overrightarrow{PQ}}$$

- The physical *line* through any two distinct points $P \triangleq (p_1, p_2)$ and $Q \triangleq (q_1, q_2)$ is represented by the mathematical line

$$l = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \mathbb{R} \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix} = \left\{ \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \lambda \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix} ; \lambda \in \mathbb{R} \right\}$$

$$\vec{c} = \left\{ (1 - \lambda) \underbrace{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}_{\vec{a}} + \lambda \underbrace{\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}}_{\vec{b}} ; \lambda \in \mathbb{R} \right\}$$

c is on line PQ



Using vectorial notation $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$ this simplifies to $l = \mathbf{p} + \mathbb{R}(\mathbf{q} - \mathbf{p}) = \{\mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}); \lambda \in \mathbb{R}\}$ and can also be written as $l = \{\lambda_1 \mathbf{p} + \lambda_2 \mathbf{q}; \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 + \lambda_2 = 1\}$.

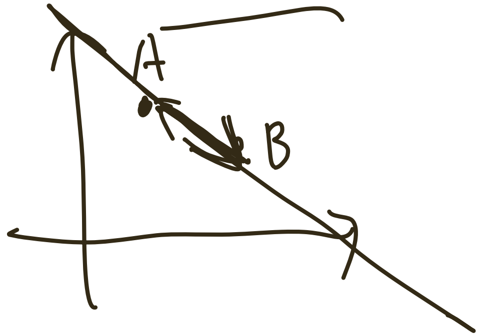
The vector $\mathbf{b} = \mathbf{q} - \mathbf{p}$ is sometimes called *direction vector* of the line l. (Note, however, that any multiple $\lambda \mathbf{b}$, $\lambda \neq 0$, and in particular $-\mathbf{b}$ is a direction vector of l as well.)

For Planes?

Think about that, and there is a related question in WS1, we will talk about that later.

Examples

Find parametric and equational representations of the line that passes through the points A (2,4,-3) and B (3,-1,1).



parametric:

$$I = \vec{A} + c\left(\vec{B} - \vec{A}\right)$$

$$I = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} + c\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$